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ON TREES OF A GRAPH AND THEIR GENERATION*

BY

S. L. HAKIMI[1]

ABSTRACT

The efficiency with which a complex electrical network may be analyzed by a digital computer depends, largely, upon the efficiency with which the trees of the corresponding graph are generated. This paper is a presentation of those properties of a set of trees that aid us in generating the trees of a given graph. Two different methods for generating trees of a graph are described. Both these methods are extremely efficient for graphs whose nullities are small in comparison to their ranks, especially if the computer is supplied with bit-wise logical instructions. An important theoretical question, that of testing a given set of trees for "completeness," is also considered.

I. INTRODUCTION

Evaluating large order determinants, even by a digital computer, is highly impractical. To avoid this difficulty, the network theorist makes use of the topological formulas by means of which he can evaluate network determinants by essentially finding all trees of corresponding graphs (**1**, **2**, **3**, **4**).[2] Several techniques for generating trees of a graph have been discovered (**5**, **6**, **7**, **8**, **9**), but none of these procedures is completely satisfactory. Some do not generate all trees (**7**); some are not easily adaptable to the digital computer (**8**).[3]

The techniques presented here have a number of advantages: (1) they give further insight into the properties of trees of a graph; (2) they can be easily programmed for a digital computer; and (3) for a large class of graphs, they will reduce the computing time necessary to generate all trees more than any of the techniques available to date.

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[1] Department of Electrical Engineering, Northwestern University, Evanston, Ill.; formerly, University of Illinois, Urbana, Ill.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

[3] Actually, Trent's (**8**) technique is the reverse process. Essentially he suggests that to find all trees of a graph one can analyze the corresponding network by determinant techniques.

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We shall make use of the concept of the distance introduced by Watanabe (7).[4] Definitions of standard terms used in this paper may be found in Seshu and Reed (10) (see also the Appendix). We shall only consider connected graphs. It is well known that this restriction involves no loss in generality (10).

II. TREES OF DISTANCE ONE FROM t_0

A subgraph g of graph G may be represented by the product of its elements. The ring sum of g_1 and g_2 denoted by $g_1 \oplus g_2 = g_3$ contains those elements of g_1 and g_2 which are not in both. The intersection of g_1 and g_2 represented[5] by $g_1 \cap g_2 = g_3$, where g_3 contains only those elements of g_1 and g_2 which are in both. Finally, the union of g_1 and g_2 is represented by $g_1 \cup g_2 = g_3$, where g_3 contains all elements of g_1 and g_2. The distance between two trees t_i and t_j is denoted by $d(t_i, t_j) = \frac{1}{2}N(t_i \oplus t_j)$, where $N(t_i \oplus t_j)$ is equal to the number of elements in the subgraph $t_i \oplus t_j$. It is easy to show that $d(t_i, t_j)$ is always a positive integer, and that

(i) $d(t_i, t_j) = d(t_j, t_i)$, for all trees t_i and t_j
(ii) $d(t_i, t_j) \leq d(t_i, t_k) + d(t_k, t_j)$, for all trees t_i, t_j, and t_k
(iii) $d(t_i, t_j) = 0$ if, and only if, $t_i = t_j$.

Therefore it is legitimate to call $d(t_i, t_j)$ a distance.

Theorem 1. An arbitrary tree t_0 and all trees of distance one from t_0 determine a graph within a 2-isomorphism.

Proof: A careful examination of Hakimi's results (11) shows that a fundamental circuit (or cut-set) matrix with respect to a tree t_0 may be obtained from a given set of trees, by merely considering all trees of distance one from t_0. Since a fundamental circuit (or cut-set) matrix determines a graph within a 2-isomorphism, this ends the proof of the theorem.

Since 2-isomorphic graphs have the same trees (10), one should be able to find all trees of a graph G from consideration of a tree t_0 and those trees of G which have a distance one from t_0.

We shall first discuss a technique for finding all trees of distance one from a given tree t_0. Let $t_0 = b_1 b_2 \cdots b_R$ where $b_1, b_2, \cdots, b_R$ are elements (branches) of t_0 and R is the rank of the graph G, and let the elements of the chord-set of t_0 be represented by $C = c_1 c_2 \cdots c_N$ where N is the nullity of G. Let the set of trees of distance i from t_0 be repre-

[4] Definition of distance given here differs slightly from Watanabe's definition.

[5] The same notation is used for the intersection of graphs as well as for the intersection of other sets.

sented by T_{0i}. Assume c_i makes a fundamental circuit with elements $b_{i1}, b_{i2}, \cdots, b_{ir}$ of t_0. Then it is clear that a tree in T_{01} may be produced by adding c_i to t_0 and removing one of the elements in $b_{i1}, b_{i2}, \cdots, b_{ir}$. Also, an arbitrary tree in T_{01} may be obtained by the above process. To see this, consider a tree t_{01} in T_{01}. t_{01} has $R - 1$ of the branches of t_0 and an element c_i of chord-set C. Let the branch of t_0 which is not in t_{01} be b_i. Adding b_i to t_{01}, we obtain a subgraph which has a circuit containing c_i. Therefore, tree t_{01} could have been generated by adding c_i to t_0 and deleting b_i from the resulting subgraph. The following theorem is a direct consequence of the above discussion.

Theorem 2: Let $B_c = [B_{c1}U]$ be the fundamental circuit matrix with respect to t_0 of a graph G. The number of trees of distance one from t_0 is equal to the number of non-zero entries in B_{c1}.

Example 1: Consider graph G, shown in Fig. 1. We wish to find all trees of distance one from $t_0 = b_1b_2b_3b_4b_5$.

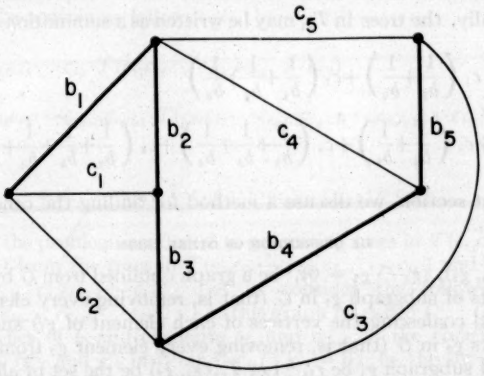

FIG. 1. A graph of rank 5 and of nullity 5.

The fundamental circuit matrix with respect to t_0 is

$$B_c = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & b_4 & b_5 & c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix} \\ \left[\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}. \qquad (1)$$

We can write all trees of distance one from t_0 in matrix form as follows (11):

$$T_{01} = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & b_4 & b_5 & c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix} \\ \left[\begin{matrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}\right] \end{array}. \tag{2}$$

Algebraically, the trees in T_{01} may be written as a summation as follows:

$$b_1b_2b_3b_4b_5\left[c_1\left(\frac{1}{b_1}+\frac{1}{b_2}\right)+c_2\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)\right.$$
$$\left.+c_3\left(\frac{1}{b_4}+\frac{1}{b_5}\right)+c_4\left(\frac{1}{b_2}+\frac{1}{b_3}+\frac{1}{b_4}\right)+c_5\left(\frac{1}{b_2}+\frac{1}{b_3}+\frac{1}{b_4}+\frac{1}{b_5}\right)\right]. \tag{3}$$

In the next section, we discuss a method for finding the other trees of G from T_{01}.

III. GENERATION OF OTHER TREES

Let $G(g_1, g_2)$, $(g_1 \cap g_2 = 0)$,[6] be a graph obtained from G by shorting all elements of subgraph g_1 in G (that is, removing every element of g_1 from G and coalescing the vertices of each element of g_1) and opening all elements g_2 in G (that is, removing every element g_2 from G). Let the rank of subgraph g_1 be r_1. Let $T_{r_1}(g_1, g_2)$ be the set of all maximal circuitless subgraphs of G containing r_1 elements of g_1 and no elements of g_2. Let the set of all trees of $G(g_1, g_2)$ be represented by $T(g_1, g_2)$, and the set of all trees of g_1 be represented by $T(0, G \oplus g_1)$.

Lemma 1: $T_{r_1}(g_1, g_2)$ is equal to the Cartesian product[7] of $T(0, G \oplus g_1)$ and $T(g_1, g_2)$. In short

$$T_{r_1}(g_1, g_2) = T(0, G \oplus g_1) \times T(g_1, g_2). \tag{4}$$

[6] 0 represents a subgraph with no elements.

[7] Cartesian product of two sets S_1 and S_2 is a set S_3 whose elements are all possible combinations of the product of an element of S_1 and an element S_2.

Proof: First we will show that every subgraph in $T_{r_1}(g_1, g_2)$ is also in $T(0, G \oplus g_1) \times T(g_1, g_2)$. Let t_{r_1} be an element in $T_{r_1}(g_1, g_2)$. Then, t_{r_1} contains r_1 elements of g_1 which together form a tree of g_1. Shorting these elements of g_1 in t_{r_1}, we obtain a tree of $G(g_1, g_2)$. This proves the first part of the theorem. Now, we must show that an element of the set $T(0, G \oplus g_1) \times T(g_1, g_2)$ is also in $T_{r_1}(g_1, g_2)$. Consider a tree $t(g_1, g_2)$ in $T(g_1, g_2)$. $t(g_1, g_2)$ is a subgraph of G in which there is no path between any of the vertices of g_1. To this subgraph one can add on a tree of g_1 and the resulting subgraph will be an element of $T_{r_1}(g_1, g_2)$.

Suppose by some procedure we have been able to find all trees of G which contains elements $c_{i_1}, c_{i_2}, \cdots, c_{i_p}$ of the chord-set and none of the remaining elements of the chord-set and also all trees containing elements $c_{i_1}, c_{i_2}, \cdots, c_{i_{p-1}}, c_{i_{p+1}}$ and none of the remaining elements of the chord-set. We wish to find all trees (if any) containing elements $c_{i_1}, c_{i_2}, \cdots, c_{i_{p+1}}$ and none of the remaining elements of the chord-set. For the moment, let us assume that the subgraph $c_{i_1}, c_{i_2}, \cdots, c_{i_{p+1}}$ is a circuitless subgraph. From Lemma 1, these three different sets of trees may be written as follows:

$$c_{i_1}c_{i_2}\cdots c_{i_p} \cdot T(c_{i_1}c_{i_2}\cdots c_{i_p}, c_{i_{p+1}}\cdots c_{i_N}) \tag{5}$$

$$c_{i_1}c_{i_2}\cdots c_{i_{p-1}}c_{i_{p+1}} \cdot T(c_{i_1}c_{i_2}\cdots c_{i_{p-1}}c_{i_{p+1}}, c_{i_p}c_{i_{p+2}}\cdots c_{i_N}) \tag{6}$$

and

$$c_{i_1}c_{i_2}\cdots c_{i_{p+1}} \cdot T(c_{i_1}c_{i_2}\cdots c_{i_{p+1}}, c_{i_{p+2}}\cdots c_{i_N}). \tag{7}$$

Therefore, the problem is reduced to generating trees in $T(c_{i_1}c_{i_2}\cdots c_{i_{p+1}}, c_{i_{p+2}}\cdots c_{i_N})$ from the trees in $T(c_{i_1}c_{i_2}\cdots c_{i_p}, c_{i_{p+1}}\cdots c_{i_N})$ and in $T(c_{i_1}c_{i_2}\cdots c_{i_{p-1}}c_{i_{p+1}}, c_{i_p}c_{i_{p+1}}\cdots c_{i_N})$. Let us consider the following graph $G(c_{i_1}c_{i_2}\cdots c_{i_{p-1}}, c_{i_{p+2}}\cdots c_{i_N})$. Let this graph be called G'. G' contains only branches of t_0 and elements c_{i_p} and $c_{i_{p+1}}$ of the chord-set. Let $T'(g'_1, g'_2)$, (g'_1 and g'_2 in G' and $g'_1 \cap g'_2 = 0$), be the set of trees of graph $G'(g'_1, g'_2)$. We can write

$$T(c_{i_1}c_{i_2}\cdots c_{i_{p+1}}, c_{i_{p+2}}\cdots c_{i_N}) = T'(c_{i_p}c_{i_{p+1}}, 0) \tag{8}$$

$$T(c_{i_1}c_{i_2}\cdots c_{i_p}, c_{i_{p+1}}c_{i_{p+2}}\cdots c_{i_N}) = T'(c_{i_p}, c_{i_{p+1}}) \tag{9}$$

and

$$T(c_{i_1}\cdots c_{i_{p-1}}c_{i_{p+1}}, c_{i_p}c_{i_{p+2}}\cdots c_{i_N}) = T'(c_{i_{p+1}}, c_{i_p}). \tag{10}$$

Therefore, our task is reduced to constructing $T'(c_{i_p}c_{i_{p+1}}, 0)$ from $T'(c_{i_p}, c_{i_{p+1}})$ and $T'(c_{i_{p+1}}, c_{i_p})$.

Let R' be the rank of G'. Let $t_i = e_{i_1}e_{i_2}\cdots e_{i_{R'-1}}$ be in $T'(c_{i_p}, c_{i_{p+1}})$ and $t_j = e_{j_1}e_{j_2}\cdots e_{j_{R'-1}}$ be in $T'(c_{i_{p+1}}, c_{i_p})$. Let the intersection $T'(c_{i_p}, c_{i_{p+1}}) \cap T'(c_{i_{p+1}}, c_{i_p}) = T'_{cc}$.

Lemma 2: (1) If t_i and t_j are not both in T'_{cc}, and if $t_i \cap t_j$ has exactly $R' - 2$ elements, then $t_i \cap t_j$ is an element of $T'(c_{i_p}c_{i_{p+1}}, 0)$. (2) Every tree in $T'(c_{i_p}c_{i_{p+1}}, 0)$ is produced by intersection of a tree in $T'(c_{i_p}, c_{i_{p+1}})$ and a tree in $T'(c_{i_{p+1}}, c_{i_p})$.

Proof: (1) Let the vertices of c_{i_p} in G' be i and j and the vertices of $c_{i_{p+1}}$ in G' be k and l. A tree in $T'(c_{i_p}, c_{i_{p+1}})$ is a circuitless subgraph (2-tree) in G' which contains $R' - 1$ elements of $G'(0, c_{i_p}c_{i_{p+1}})$ and no paths between vertices i and j. Similarly a tree in $T'(c_{i_{p+1}}, c_{i_p})$ is a circuitless subgraph (2-tree) in G' which contains $R' - 1$ elements of $G'(0, c_{i_p}c_{i_{p+1}})$ and no paths between vertices k and l. Let t_i be in $T'(c_{i_p}, c_{i_{p+1}})$ but not in T'_{cc}. Then, t_i is a circuitless subgraph in G' which has no paths from i to j, but has a path from k to l. For if t_i also had no path from k to l, then t_i would be in T'_{cc} (12). Let us assume $t_i \cap t_j$ contains $R' - 2$ elements. Subgraph $t_i \cap t_j$ in G' has no paths either from vertex i to j or from vertex k to l. To establish that $t_i \cap t_j$ is in $T'(c_{i_p}c_{i_{p+1}}, 0)$, we must also show that if subgraph $t_i \cap t_j$ has a path from, say, vertex i to, say, vertex k, then $t_i \cap t_j$ has no path from vertex j to l. However, if one examines t_i, one can see that t_i has the same property, therefore, $t_i \cap t_j$ must have this property, also. This proves the first part of the lemma. (2) Let t_p be in $T'(c_{i_p}c_{i_{p+1}}, 0)$. We must show that there exist two elements in $G'(0, c_{i_p}c_{i_{p+1}})$, say, b_i and b_j such that t_pb_i is in $T'(c_{i_p}, c_{i_{p+1}})$ and t_pb_j is in $T'(c_{i_{p+1}}, c_{i_p})$, and furthermore, not both of the subgraphs t_pb_i and t_pb_j are in T'_{cc}. Suppose there exists no such a pair of elements. Either (1) addition of every branch b_i in $G'(0, c_{i_p}c_{i_{p+1}})$ to t_p forms a circuit with t_p which implies that there are no trees either in $T'(c_{i_p}, c_{i_{p+1}})$ or in $T'(c_{i_{p+1}}, c_{i_p})$; or (2) addition of every branch b_i in $G'(0, c_{i_p}c_{i_{p+1}})$ to t_p gives a subgraph in T'_{cc} which implies that $G'(0, c_{i_p}c_{i_{p+1}})$ is not connected. Both of these are contradictions to the original assumptions, hence the lemma.

At this point, one must note that if $c_{i_1}c_{i_2}\cdots c_{i_{p+1}}$ contained a circuit, then elements c_{i_p} and $c_{i_{p+1}}$ would be in parallel in G' (that is, they would have the same vertices). This in turn would imply that $T'(c_{i_p}, c_{i_{p+1}}) = T'(c_{i_{p+1}}, c_{i_{p+1}})$. According to Lemma 2, the set $T'(c_{i_p}c_{i_{p+1}}, 0) = 0$. Therefore, we will not have to test independently to see whether or not $c_{i_1}c_{i_2}\cdots c_{i_{p+1}}$ is circuitless. For if it contains a circuit, then automatically $T'(c_{i_p}c_{i_{p+1}}, 0) = 0$, which is what is desired to make Eq. 8 correct regardless of the fact that $c_{i_1}c_{i_2}\cdots c_{i_{p+1}}$ contains a circuit or not.

Lemma 2 suggests a way to generate all trees of a graph from the trees of distance one from a given tree t_0. Trees of distance two are

found by considering (as suggested by Lemma 2) all combinations of sets of trees of distance one. For example, the set of trees in $T(c_1c_2, c_3c_4 \cdots c_N)$ are found from a set of trees $T(c_1, c_2c_3 \cdots c_N)$ and $T(c_2, c_1c_3 \cdots c_N)$. This operation is similarly continued until all trees in T_{02} are found. To find trees in T_{03} we do the same, and so on.

Example 2: Suppose we want to find all trees of a graph of Fig. 2 by means of the process just described.

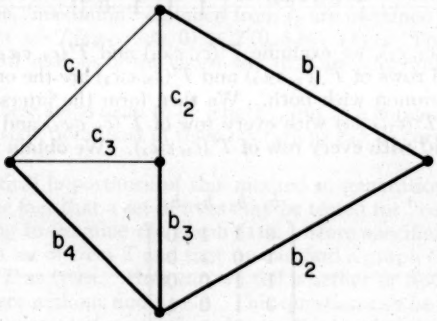

FIG. 2. A graph of rank 4 and nullity 3.

Let us start with tree $t_0 = b_1b_2b_3b_4$. Trees in T_{01} may be given in the form of a matrix as follows:

$$T_{01} = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & b_4 & c_1 & c_2 & c_3 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}. \tag{11}$$

From T_{01}, we see that

$$T'(c_1, c_2c_3) = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \tag{12}$$

$$T'(c_2, c_1c_3) = \begin{matrix} \\ \\ \\ \end{matrix} \begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \tag{13}$$

and

$$T'(c_3, c_1c_2) = \begin{matrix} \\ \\ \end{matrix} \begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \tag{14}$$

To find $T'(c_1c_3, c_2)$, we examine $T'(c_1, c_2c_3)$ and $T'(c_2, c_1c_3)$ and we see that the third rows of $T'(c_1, c_2c_3)$ and $T'(c_2, c_1c_3)$ are the only rows that are not in common with both. We then form the intersection of the third row of $T'(c_1, c_2c_3)$ with every row of $T'(c_2, c_1c_3)$ and third row of $T'(c_2, c_1c_3)$ and with every row of $T'(c_1, c_2c_3)$. We obtain the following matrix:

$$\begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}. \tag{15}$$

We see that the third row and sixth row of the above matrix are identical. Therefore, omitting one of these two identical rows, we obtain

$$T'(c_1c_2, c_3) = \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{16}$$

Going through exactly the same process, we find that $T'(c_1c_3, c_2) = T'(c_2c_3, c_1) = T'(c_1c_2, c_3)$. This automatically proves that there exists no tree containing elements $c_1c_2c_3$. Looking at the graph we see that $c_1c_2c_3$ indeed forms a circuit. Trees in T_{0_2} may be written in the form of an algebraic sum as follows:

$$T(c_1c_2, c_3) \times (c_1c_2 + c_2c_3 + c_1c_3). \tag{17}$$

We have thus found all trees from the set of trees in T_{0_1}. Although this technique is undoubtedly a better technique for generating

trees of a graph than merely for considering all possible combinations of R elements of G, it has many undesirable features, some of which are: (1) one must consider almost all possible combinations of the chord-set $(2^N - N)$ which could be an extremely large number; (2) in the process of finding trees of a graph by this method, a tree may be generated more than once. Therefore, one must examine every set of new trees which is obtained by the process of Lemma 2 within itself to eliminate duplication. The first of these undesirable features may be remedied (to an extent) by first finding all trees in $T(0, b_1b_2\cdots b_R)$. From Lemma 1, all trees of the "maximum" distance from t_0 are obtained by the Cartesian product of $T(c_1c_2\cdots c_N, 0) \times T(0, b_1b_2\cdots b_R)$. Then, consider only those combinations of the chord-sets that are subgraphs of trees in $T(0, b_1\cdots b_R)$.

Many of these steps seem to be lengthy, but in a computer with bitwise logical instructions most of these operations can be carried out with great speed.

The theoretical importance of this method of generation of a set of trees lies in the fact that a set of trees may be tested for "completeness" without having to examine the graph (11). More specifically, suppose we are given a set of trees T and that we can find a graph G that has all subgraphs in T as trees. How can we tell whether or not G will have some other trees without finding G? This question can be answered by the use of the process of generation of trees discussed above, as follows: Pick an arbitrary tree t_0 in T. Pick all trees in T that are of distance one from t_0. From this new set, generate all other trees. Only if the set thus produced is identical with T can we say that T is a complete set of trees.

IV. AN ALTERNATIVE PROCEDURE

Let $B_c = [B_{c_1}U]$ be a fundamental circuit matrix of a graph G with respect to a tree t_0. We want to find all trees containing elements $c_{i_1}c_{i_2}\cdots c_{i_p}$ $(p \leq N)$ of the chord-set, if $c_{i_1}c_{i_2}\cdots c_{i_p}$ is a circuitless subgraph of G. From Lemma 1, all such trees may be written as

$$c_{i_1}c_{i_2}\cdots c_{i_p} \times T(c_{i_1}c_{i_2}\cdots c_{i_p}, c_{i_{p+1}}c_{i_{p+2}}\cdots c_{i_N}) \tag{18}$$

where, as before $T(c_{i_1}\cdots c_{i_p}, c_{i_{p+1}}\cdots c_{i_N})$ is the set of all trees of the graph $G(c_{i_1}\cdots c_{i_p}, c_{i_{p+1}}\cdots c_{i_N}) = G_1$. The first question to answer is: How can we find a fundamental circuit matrix of G_1 from B_c?

Let a matrix obtained by taking the rows $i_1, i_2, \cdots, i_p$ of B_{c_1} be represented by B_1. It is easy to see that the rows of B_1 represent circuits in G_1. We want to show that every circuit in G_1 is a linear combination (mod. 2) of rows of B_1. More precisely, we want to prove the following theorem.

Theorem 3: The rank of B_1 (mod. 2) is equal to the nullity of G_1.

Proof: Suppose the theorem is not correct, then there exists a circuit c in G_1 which cannot be obtained as a linear combination of rows of B_1. The elements of c are branches of t_0 in G. From necessity, elements in c and some combination of elements $c_{i_1}c_{i_2}\cdots c_{i_p}$ must form a circuit c_p in G. However, we know that circuit c_p can be found as a linear combination of rows of B_c. This proves that exactly the same linear combination of rows of B_1 must form circuit c. Hence the theorem.

Theorem 4: Let rank of B_1 be equal to n_1. $c_{i_1}c_{i_2}\cdots c_{i_p}$ is a circuitless subgraph if, and only if, $n_1 = p$.

Proof: (1) If $n_1 < p$, then some subset of rows of B_1 when added (mod. 2) together results in a row of zeros. Adding the corresponding rows of B_c we obtain a row matrix B'_1 which has only non-zero entries in some of the columns corresponding to element $c_{i_1}, c_{i_2}, \cdots, c_{i_p}$. Since every linear combination of rows of B_c is either a circuit or a union of disjoint circuits, we can conclude that B_1' corresponds to a circuit or a union of disjoint circuits; hence, $c_{i_1}c_{i_2}\cdots c_{i_p}$ contains a circuit. (2) Suppose $c_{i_1}c_{i_2}\cdots c_{i_p}$ contains a circuit, $c = c_{i_{j_1}}c_{i_{j_2}}\cdots c_{i_{j_q}}$. Adding rows $i_{j_1}, i_{j_2}, \cdots, i_{j_q}$ of B_c we must obtain a row matrix B'_2 which has only non-zero entries in the columns corresponding to the elements $c_{i_{j_1}}, c_{i_{j_2}}\cdots c_{i_{j_q}}$. Therefore, addition of corresponding rows of B_1 would result in a row of zeros. Hence rank B_1 is less than p.

Therefore, if $r_1 < p$ there exist no trees containing elements $c_{i_1}, c_{i_2}, \cdots, c_{i_p}$. If $r_1 = p$ then all trees containing $c_{i_1}, c_{i_2}, \cdots, c_{i_p}$ may be found by Eq. 18 and by merely finding all trees of G_1. However, we know a circuit matrix of G_1 is B_1. By successive row operations on B_1, we can obtain a fundamental circuit matrix $B_c^{(1)}$ of G_1. We can find the trees of G_1 by repeating the entire process on $B_c^{(1)}$. Therefore, essentially, the problem of finding trees of a large graph may be reduced to the problem of finding the trees of a number of reduced graphs.

Example 3: Let us find trees of the graph of Fig. 2 by the process just described. The fundamental circuit matrix of the graph of Fig. 2 is

$$B_c = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & b_4 & c_1 & c_2 & c_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

The trees of distance one from $t_0 = b_1b_2b_3b_4$ are found in exactly the same way as discussed in Section II. To find the trees containing elements c_1c_2, we examine the following matrix, which is a circuit matrix of the graph $G(c_1c_2, c_3)$

$$B_1 = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Clearly the rank of B_1 is equal to two. A tree of $G(c_1c_2, c_3)$ is b_1b_2, the trees of $G(c_1c_2, c_3)$ of distance one from b_1b_2 are b_1b_4, b_2b_4, b_1b_3, b_2b_3. To see whether or not $G(c_1c_2, c_3)$ contains a tree having both the elements b_3 and b_4, we examine the circuit matrix of $G(c_1c_2b_3b_4, c_3)$

$$B_1^{(1)} = \begin{array}{c} \begin{matrix} b_1 & b_2 \end{matrix} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}.$$

However the rank of $B_1^{(1)}$ is less than the number of rows of $B_1^{(1)}$. Hence, $G(c_1c_2, c_3)$ contains no tree having elements b_3 and b_4. The trees of G containing c_1c_3 and c_2c_3 are found by similar consideration of the circuit matrices of the graph $G(c_1c_3, c_2)$ and $G(c_2c_3, c_1)$. To see whether or not there exist any trees containing elements $c_1c_2c_3$, we examine the circuit matrix of $G(c_1c_2c_3, 0)$.

$$B_2 = \begin{array}{c} \begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

It is clear that the rank of the above matrix is equal to 2. Therefore there exist no trees in G containing elements $c_1c_2c_3$.

Unfortunately, this procedure suffers from the same disadvantages as the first procedure. However, here, there will be no duplications.

V. CONCLUSIONS

The two methods of generating trees of a graph presented here have undesirable features, some of which can be rectified by an efficient program on a digital computer supplied with bit-wise logical instructions. Unfortunately the other disadvantages are more complex in nature and require further study of properties of trees of linear graphs. In general, both these methods are very efficient for graphs with "large" ranks and "small" nullities. As intuitively suspected, the geometry of the graph will play an important role in the choice of the method. Mayeda's technique (5) later programmed by Kim, Younger, Freiman, and Mayeda (6) provides an efficient first step for a graph which has a cut-set containing two or three elements. The result of Lemma 1 seems to suggest the existence of more efficient techniques for generation of trees of a graph; however, the author was unable to make better

use of it. It is our hope that the results presented in this paper may be used to find still more efficient methods for finding trees of a linear graph.

Many of the results presented in this paper have other theoretical significance (that of completeness was specifically pointed out). Lemma 2, for example, suggests how to find "3-trees" (or trees in $T(c_1c_2, 0)$) from two sets of "2-trees" (or trees in $T(c_1, c_2)$ and $T(c_2, c_1)$. This result is of importance in the topological considerations in the design of 2-port networks (10).

Acknowledgment

The help of Professors M. E. Van Valkenburg and W. Mayeda of the University of Illinois in the preparation of this paper is gratefully acknowledged.

REFERENCES

(1) W. MAYEDA AND M. E. VAN VALKENBURG, "Network Analysis and Synthesis by Digital Computer," IRE Wescon Convention Record (part 2), 1957, pp. 137–144.
(2) W. MAYEDA AND M. E. VAN VALKENBURG, "Analysis of Non-Reciprical Networks by Digital Computers," IRE Wescon Convention Record (part 2), 1958, pp. 70–75.
(3) F. J. MACWILLIAMS, "Topological Network Analysis as a Computer Program," *IRE Trans. on Circuit Theory*, Vol. CT-5, pp. 228–229, September, 1958.
(4) E. W. HOBBS, "Topological Network Analysis as a Computer Program (Discussion)," *IRE Trans. on Circuit Theory*, Vol. CT-6, pp. 135–136, March, 1959.
(5) W. MAYEDA, "Reducing Computation Time in the Analysis of Networks by Digital Computer," *IRE Trans. on Circuit Theory*, Vol. CT-6, pp. 136–137, March, 1959.
(6) W. H. KIM, D. H. YOUNGER, C. V. FREIMAN AND W. MAYEDA, "On Iterative Factorization in Network Analysis by Digital Computers," Eastern Joint Computer Conference, December, 1960.
(7) H. WATANABE, "A Computational Method for Network Topology," *IRE Trans. on Circuit Theory*, Vol. CT-7, pp. 296–302, September, 1960.
(8) H. M. TRENT, "Note on the Enumeration and Listing of all Possible Trees in a Connected Linear Graph," *Proc. Natl. Acad. Sci. U. S.*, pp. 1004–1007, October, 1954.
(9) N. NAKAGAWA, "On Evaluation of Graph Trees and the Driving-Point Admittances," *IRE Trans. on Circuit Theory*, Vol. CT-5, pp. 122–127, June, 1958.
(10) S. SESHU AND M. B. REED, "Linear Graphs and Electrical Networks," Reading, Mass., Addison-Wesley, 1961.
(11) S. L. HAKIMI, "On Realizability of a Set of Trees," *IRE Trans. on Circuit Theory*, Vol. CT-8, pp. 11–18, March, 1961.
(12) S. L. HAKIMI, "Graphs with Two Kinds of Elements," JOUR. FRANKLIN INST., Vol. 270, pp. 451–467 (1960).

APPENDIX

The definitions of some of the terms used in this paper:

A *tree* of a graph is a subgraph with a maximum number of elements that has no circuits (loops).

A *chord-set* of a tree is those elements of the graph that are not in that tree.

Rank R of a graph is the number of elements in a tree of the graph, and the *nullity* N of a graph is the number of elements in a chord-set.

A fundamental circuit with respect to a tree t_0 is a circuit containing only one element of the chord-set of t_0. Clearly, there are N such circuits. A representation of these N fundamental circuits in the form of a matrix in which rows correspond to the circuits and the first R columns correspond to the elements of t_0 and the last N columns correspond to elements of the chord-set of t_0, is called a fundamental circuit matrix with respect to tree t_0 and is written as $B_e = [B_{e_1}\ U]$ where U is an identity matrix of order N.

A *cut-set* is a minimal set of elements whose removal from the graph reduces the rank of the graph by one.

A *fundamental cut-set* with respect to a tree t_0 is a cut-set containing only one element of t_0. Clearly, there are R fundamental cut-sets with respect to t_0. Similarly, a fundamental cut-set matrix is a matrix representative of these R cut-sets.

Two graphs G_1 and G_2 having the same circuits are called 2-isomorphic, and it is said that G_1 and G_2 are identical within a 2-isomorphism.

A GENERALIZATION OF THE PHASE RELATIONS IN A FORCED HARMONIC OSCILLATOR*

BY

B. A. FLEISHMAN[1]

ABSTRACT

When a simple harmonic oscillator is subjected to a sinusoidally varying external force whose frequency is different from the natural frequency of the oscillator, there is a simple and well-known phase relation between the purely forced response and the forcing function. They will have the same or opposite signs (except when both functions vanish) according as the forcing frequency is less or greater than the natural frequency.

It is shown here that when the sinusoidal forcing function is replaced by any member of a wide class of periodic functions, the relation between the signs of the forcing function and the forced response continues to hold when the forcing frequency is greater than the natural frequency, but no longer holds in the other case.

INTRODUCTION

Suppose a harmonic oscillator, undamped and with natural frequency one, is subjected to a sinusoidally varying external force with frequency $\omega \neq 1$. The governing differential equation is

$$\frac{d^2x}{dt^2} + x = b \sin \omega t,$$

where b is an arbitrary constant. The particular solution

$$x(t) = \frac{b}{1 - \omega^2} \sin \omega t$$

represents the forced component in the response. It is evident from inspection that the forced oscillation $x(t)$ is in phase, or 180° out of phase, with the forcing function $b \sin \omega t$, according as $\omega < 1$ or $\omega > 1$. To put it another way: except at the instants $t = m\pi/\omega$, $m = 0, \pm 1, \pm 2, \cdots$, where they both vanish, these two functions always have the same sign, or opposite signs, according as ω is less or greater than 1. This result is well known.

In this note, for more general periodic forcing functions $f(t)$, we obtain an analogous result about the signs of the forcing function and

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[1] Rensselaer Polytechnic Institute, Troy, N. Y.; presently on leave at the U. S. Army Mathematics Research Center, University of Wisconsin, Madison, Wis.

the forced oscillation when ω, the frequency of $f(t)$, is greater than 1, and we show that there is no analogue in the case $\omega < 1$. These questions arose during an investigation extending the author's previous results[2] on the periodic responses of an on-off control system to periodic inputs.

GENERALIZATION OF THE SIGN RELATIONSHIP IN THE CASE $\omega > 1$

Consider the differential equation

$$\frac{d^2x}{dt^2} + x = f(t) \tag{1}$$

where the function $f(t)$ is continuous for all t. Suppose further that $f(t)$ may be represented for all t by a Fourier sine series of the form

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t, \tag{2}$$

where the Fourier coefficients b_n are given by the usual formula. Then $f(t)$ is periodic with period $2\pi/\omega$, and $f(m\pi/\omega) = 0$, $m = 0, \pm 1, \pm 2, \cdots$.

Finally, let $f(t)$ be of constant sign in the interval I_1: $0 < t < \pi/\omega$. Since

$$\sin n\omega\left(\frac{2\pi}{\omega} - t\right) = -\sin n\omega t \qquad (n = 1, 2, \cdots)$$

it follows from (2) that $f(2\pi/\omega - t) = -f(t)$. Therefore, whatever the sign of $f(t)$ in I_1, it has the opposite sign in I_2: $\pi/\omega < t < 2\pi/\omega$, and continues to alternate in sign because of the periodicity. It is always assumed that $\omega \neq 1$.

We shall denote by $x_f(t)$ the solution of (1) which has a continuous second derivative everywhere and is periodic, with period $2\pi/\omega$. It will be seen that whenever $\omega > 1$, there exists a unique $x_f(t)$; it represents, of course, the purely forced response of the oscillator to the excitation $f(t)$.

Theorem. If $\omega > 1$, $x_f(t)$ is exactly 180° out of phase with $f(t)$; that is, $f(t)$ and $x_f(t)$ are of opposite sign for all values of t except $t = m\pi/\omega$, $m = 0, \pm 1, \pm 2, \cdots$, where both functions vanish.

To prove first that $x_f(t)$ is unique, suppose there exist two functions, $x_f(t)$ and $\bar{x}_f(t)$, which are twice continuously differentiable everywhere, have period $2\pi/\omega$, and satisfy the nonhomogeneous linear differential

[2] B. A. FLEISHMAN, "Harmonics and Subharmonic Response of an On-Off Control System to Sinusoidal Inputs," JOUR. FRANKLIN INST., Vol. 270, pp. 99–113 (1960).

equation (1). Their difference, $\xi(t) = x_f - \bar{x}_f$, must then satisfy the homogeneous equation $d^2\xi/dt^2 + \xi = 0$, and is therefore expressible in the form $A \cos(t + \phi)$, where A and ϕ are integration constants. On the other hand, $\xi(t)$, being the difference of two periodic functions, has itself the period $2\pi/\omega$, where $\omega > 1$. The only function of the form $\xi(t) = A \cos(t + \phi)$, however, with period $2\pi/\omega$ is $\xi(t) \equiv 0$. Therefore there exists at most one function $x_f(t)$.

The existence of $x_f(t)$ is established by considering one form of $x_f(t)$ which suggests itself immediately, namely, the Fourier series

$$x_f(t) = \sum_{n=1}^{\infty} \frac{b_n}{1 - n^2\omega^2} \sin n\omega t. \tag{3}$$

This series, it can be shown, is a twice continuously differentiable solution of (1) when $f(t)$, besides being continuous, has a piecewise continuous first derivative. It is evident too that the right member of (3) has period $2\pi/\omega$, and it also has the property $x_f(2\pi/\omega - t) = -x_f(t)$. Therefore, if $x_f(t)$ has a constant sign throughout the interval I_1, it will have the opposite sign during the next half-period, I_2, and keep alternating in sign in successive half-periods, as does $f(t)$. The question reduces then to a determination of the sign of $x_f(t)$ on I_1; for this purpose, however, the representation (3) will not do. Essentially the reason is that while corresponding coefficients in (2) and (3) are of opposite sign (since $\omega > 1$), the magnitudes of the coefficients have changed in varying proportions.

We introduce, then, an alternative representation of $x_f(t)$,

$$x_f(t) = \left(\sin\frac{\pi}{\omega}\right)^{-1}\left[\sin\left(t - \frac{\pi}{\omega}\right)\int_0^t f(\tau)\sin\tau d\tau + \sin t \int_t^{\pi/\omega} f(\tau)\sin\left(\tau - \frac{\pi}{\omega}\right)d\tau\right], \tag{4}$$

obtained by the method of variation of parameters (for example, see Rainville[3]). It is easily verified by direct substitution that when $f(t)$ is continuous, the right-hand member of (4) has a continuous second derivative and satisfies the differential equation (1). It may also be shown, by straightforward (but by no means trivial) computations, that this function possesses the properties $x(2\pi/\omega + t) = x_f(t)$ and $x_f(2\pi/\omega - t) = -x_f(t)$. Now since $\pi/\omega < \pi$, $\sin(\pi/\omega) > 0$; also, for $0 < t < \pi/\omega$, $\sin t > 0$ and $\sin\tau \geqslant 0$ in (4). Similarly $\sin(t - \pi/\omega) < 0$ and $\sin(\tau - \pi/\omega) \leqslant 0$. Examination of (4) then reveals that if $f(t) > 0$ on I_1, $x_f(t) < 0$ there, and vice versa, which completes the proof of the theorem.

[3] E. D. RAINVILLE, "Elementary Differential Equations," New York, The Macmillan Co., 1952.

It should be noted that the conclusion about the sign of $x_f(t)$ on I_1 does not require that $f(t)$ be of constant sign there; the weaker assumption that $f(t)$ is non-negative (but not identically zero) on I_1 also yields the result that $x_f(t) < 0$ there.

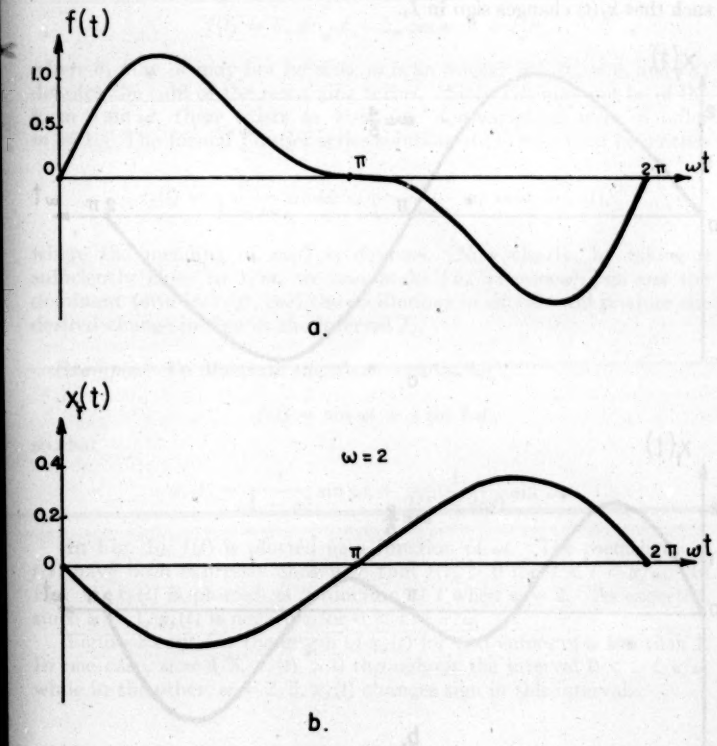

FIG. 1.

IMPOSSIBILITY OF GENERALIZATION IN THE CASE $\omega < 1$

We recall that when $f(t) = b \sin \omega t$ and $\omega \neq 1$, $x_f(t) = [b/(1-\omega^2)] \sin \omega t$, so that except at $t = m\pi/\omega$, $m = 0, \pm 1, \pm 2, \cdots$, where they both vanish, the functions $f(t)$ and $x_f(t)$ always agree in sign if $\omega < 1$ and disagree in sign if $\omega > 1$. The theorem proved above states that, when $\omega > 1$, this sign relationship holds for any continuous periodic forcing function

$f(t)$ which may be represented by a series of the form (2) and which has a fixed sign in I_1. We wish now to show that there is no corresponding generalization in the case $\omega < 1$. Indeed, if we take any given $f(t)$ of the type specified (excluding simple sinusoids $b \sin \omega t$) and allow its frequency ω to vary, then there is always a value of ω between 0 and 1 such that $x_f(t)$ changes sign in I_1.

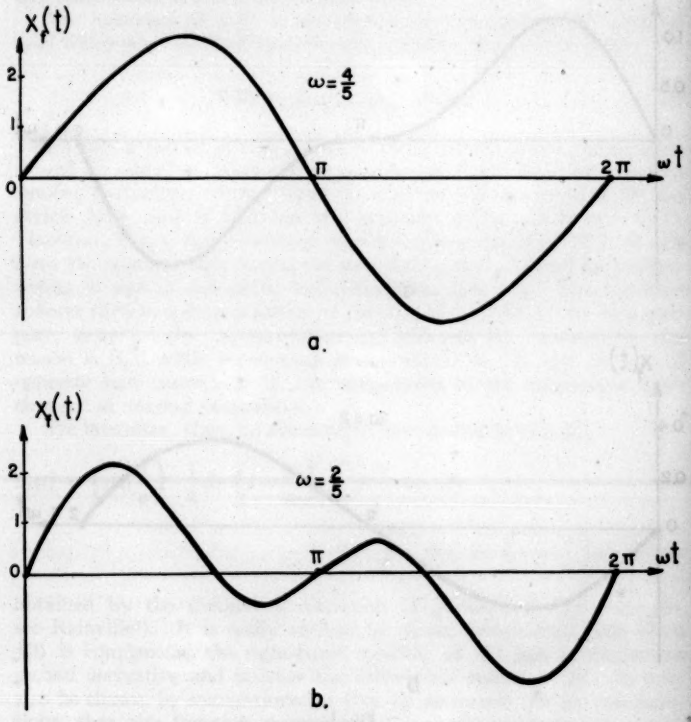

FIG. 2.

The crux of the argument is the possibility of resonance. The Fourier expansion of such a function of period $2\pi/\omega$ will have components of frequency $n\omega$, where n takes on one or more of the values $2, 3, 4, \cdots$. If ω happens to be the reciprocal of an integer m, and the term $n = m$ occurs in the Fourier expansion of $f(t)$, then resonance will occur, and there will be no periodic response $x_f(t)$. Even if we exclude

the values $\omega = 1/n$, $n = 2, 3, \cdots$ from consideration, however, in order to guarantee the existence of $x_f(t)$, the required change in sign may still be effected by taking ω sufficiently close to the reciprocal of some integer.

Specifically, let us write $f(t)$ in the form

$$f(t) = b_1 \sin \omega t + b_m \sin m\omega t + r(t)$$

where b_1 may or may not be zero, m is an integer $\neq 1$, $b_m \neq 0$, and $r(t)$ denotes the sum of the remaining terms. Since $f(t)$ may not be of the form $b \sin \omega t$, there exists at least one non-vanishing term of order $m \neq 1$. The formal Fourier series solution of (1) may then be written

$$x_f(t) = \frac{b_1}{1 - \omega^2} \sin \omega t + \frac{b_m}{1 - m^2\omega^2} \sin m\omega t + x_r(t),$$

where the meaning of $x_r(t)$ is obvious. Now clearly, by taking ω sufficiently close to $1/m$, we can make $[b_m/(1 - m^2\omega^2)] \sin m\omega t$ the dominant term in $x_f(t)$, and the oscillations in $\sin m\omega t$ will produce the desired change in sign in the interval I_1.

Example. To illustrate the above results, let

$$f(t) = \sin \omega t + \tfrac{1}{2} \sin 2\omega t,$$

so that

$$x_f(t) = \frac{1}{1 - \omega^2} \sin \omega t + \frac{1}{2(1 - 4\omega^2)} \sin 2\omega t.$$

In Fig. 1*a*, $f(t)$ is plotted as a function of ωt. The coefficients in $f(t)$ have been expressly chosen so that $f(t) > 0$ for $0 < t < \pi/\omega$. In Fig. 1*b*, $x_f(t)$ is plotted as a function of t when $\omega = 2$. As expected, since $\omega > 1$, $x_f(t)$ is negative for $0 < t < \pi/\omega$.

Figure 2 contains the graph of $x_f(t)$ for two values of ω less than 1. In one case, $\omega = 4/5$, $x_f(t) > 0$ throughout the interval $0 < t < \pi/\omega$, while in the other, $\omega = 2/5$, $x_f(t)$ changes sign in this interval.

LINEARLY SEPARABLE SWITCHING FUNCTIONS

BY

C. L. COATES[1] AND P. M. LEWIS II[1]

ABSTRACT

A threshold switching element is a multi-input generalization of an "and" or "or" gate. Its output is unity if and only if the weighted sum of its inputs equals or exceeds a threshold

$$\Sigma a_i x_i \geq T;$$

otherwise its output is zero. A linearly separable function is a switching function that can be realized with one such element.

This paper studies in some detail the properties of linearly separable functions. In terms of these properties a test and realization procedure is derived. The test consists of three parts which for every switching function provide either a realization or a proof that the function is not linearly separable. Not all parts are necessarily required for testing every function. A function of eight variables can be tested and realized in about thirty minutes of hand computations.

I. INTRODUCTION

This paper is concerned with the realization of logical or switching functions by means of threshold components. A *threshold component*, illustrated in Fig. 1, has n binary inputs $x_1, x_2, \cdots, x_n$ and a single

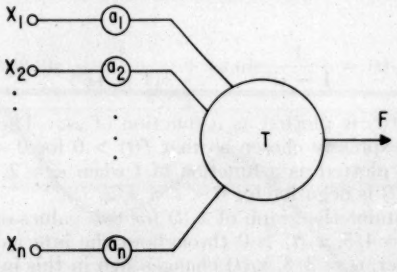

FIG. 1. A threshold component.

binary output. It has an internal *threshold* T and internal *weights* or *coefficients* $a_1, a_2, \cdots, a_n$ such that a_k is associated with x_k. The threshold and the coefficients can have any real value. The component operates as follows:

If p_j denotes one of the combinations of values which the input variables can take on, and if $f(p_j)$ denotes the weighted sum $\sum_{k=1}^{n} a_k x_k$

[1] General Electric Research Laboratory, Schenectady, N. Y.

corresponding to p_j, then

$$f(p_j) \geq T \Rightarrow \text{output} = 1 \Rightarrow F(p_j) = 1$$
$$f(p_j) < T \Rightarrow \text{output} = 0 \Rightarrow F(p_j) = 0$$

where F denotes the switching function realized by the threshold component.

This paper presents a test and realization procedure which for every given switching function provides either a realization of that function in terms of a single threshold component or a proof that its realization requires more than one such component. The test consists of three parts; however, all parts are not necessarily required to test every function.

At the time this work was initiated the report by McNaughton (4)[2] was available. Since that time additional results have appeared and those of which we are aware are listed in the references.

The main part of this paper is divided into two parts: Section II is concerned with the properties of logical functions that can be realized by a single threshold component; Section III is a description of the test and realization procedure.

In order to reduce the length of this paper much of the material is presented without proof and with a minimum of explanatory discussion. A more complete presentation which includes the proofs of all properties as well as additional examples and discussion is given in a report (2) to which frequent reference is made.

II. PROPERTIES OF LINEARLY SEPARABLE FUNCTIONS

A. *Definitions*

A *switching function* F is any Boolean function of binary variables $x_1, \cdots, x_n$. Each combination of values which may be assigned to the variables of F is called a point of F and is denoted by p_j. The collection $\{p_j\}$, called the *points of* F is the set on which F is defined.

A *plane function*, denoted by f, is any function of the form $\sum_{k=1}^{n} a_k x_k$ which is defined on $\{p_j\}$ where the x_k are binary variables, the a_k are real constants and where the algebraic operations are the usual ones (non-Boolean). The equation

$$f = \sum_{k=1}^{n} a_k x_k = T$$

is called a *plane equation* of F.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

A switching function F is called *linearly separable* if there is a plane function f and a real number T, such that $f(p_j) \geqslant T$ when $F(p_j) = 1$ and $f(p_j) < T$ when $F(p_j) = 0$.

From this definition and the description of threshold components given in the Introduction, the following property is evident.

PROPERTY 1. *Any linearly separable switching function can be realized by a single threshold component.*

Although it is easy to give an n-dimensional geometric interpretation of linearly separable switching functions, we have found the following alternative interpretation more convenient for purposes of both analysis and visualization. Inasmuch as the real function f and the Boolean function F are defined on $\{p_j\}$, then to each p_j there corresponds the ordered pair $f(p_j)$, $F(p_j)$ where $f(p_j)$ is some real number and where $F(p_j) = 0, 1$. The set $\{f(p_j), F(p_j)\}$ of ordered pairs of numbers is called a *map* of F. A convenient visualization for a map is shown in Fig. 2 where for each ordered pair $f(p_j)$, $F(p_j)$ a point is plotted on the

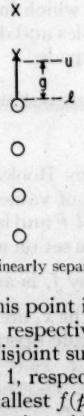

FIG. 2. The map of a linearly separable switching function.

real line at location $f(p_j)$ and this point is represented by 0 or X according to whether $F(p_j) = 0$ or 1, respectively.

A map is divided into two disjoint subsets, called *zero* and *unit parts*, according to whether $F = 0$ or 1, respectively. Of the members of the unit part, let u denote the smallest $f(p_j)$. If the unit part is empty, as is the case of switching functions identically zero, let $u = \infty$. Of the members of the zero part, let l denote the largest $f(p_j)$. If the zero part is empty, as is the case of switching functions identically one, let $l = -\infty$. A map is called *separated* if $u > l$.

A switching function F is linearly separable if and only if it has at

least one separated map. A plane function which defines a separated map is called a *separating plane function.* The *gap*, G, of a separated map is the set of all real numbers y such that $l < y \leqslant u$. The gap is specified by the notation $u:l$ where u and l are called, respectively, the *upper* and *lower boundaries* of the gap. The *gap length*, g, is given by $g = u - l$. In the realization of a linearly separable switching function the threshold can be anywhere within the gap.

Switching functions which are identically 0 or identically 1 are called *constant functions*, and are denoted by $F \equiv 1$ or $F \equiv 0$, respectively. Such functions are linearly separable and any plane function is a separating plane function. In fact, for separating plane function

$$f = \sum_{k=1}^{n} a_k x_k,$$

$F \equiv 1$ has gap $0: -\infty$ and $F \equiv 0$ has gap $\infty : \sum_{a_k > 0} a_k$.

B. Properties from Coefficient Sign

There are a number of properties of linearly separable functions relating the sign of the coefficient, a_k, in the separating plane function to the appearance of the variable, x_k, in the minimum sum of products (min-term) form of F. These properties were originally derived by McNaughton (4) and are listed below without proof.

PROPERTY 2. *If a linearly separable function is expressed in min-term form, no variable can appear both complemented and uncomplemented. A function with this property is called a* unate function; *thus all linearly separable functions are unate.*

PROPERTY 3. *If a variable appears uncomplemented in the min-term expression of a linearly separable function, its coefficient in the separating plane function is positive. If a variable appears complemented in the min-term expression of a linearly separable function, its coefficient in the separating plane function is negative.*

PROPERTY 4. *A unate function is linearly separable if and only if the function obtained from the min-term form by uncomplementing all of the complemented variables is linearly separable. Moreover, the separating plane equations of the two functions are simply related as follows: If a complemented variable is uncomplemented, the sign of its coefficient is changed from negative to positive and the magnitude of that coefficient is added to the threshold. The size of the gap remains the same.*

Property 4 provides a means whereby every linearly separable function can be converted to a positive variable function which is "*equivalent*" for purposes of testing and realization. For the remainder of this paper, therefore, we confine our attention to functions with only positive (that is, uncomplemented) variables. Therefore, *unless stated otherwise, all properties and proofs thereof are confined to the positive variable case.*

The *relative gap*[3] of a realization is defined in terms of the realization of the positive variable "equivalent" as the ratio of the gap length to the threshold when the threshold is selected as the midpoint of the gap. Relative gap is a measure of the reliability of the component where reliability refers to the sensitivity of the operation of the component to changes in the values of the coefficients and the threshold. In terms of this measure the most reliable realization is one with the largest relative gap.

C. *Properties from Coefficient Size*

a. *Reduced Functions*

A function F_1 *contains* a function F_2, written $F_1 \geqslant F_2$, if $F_1(p_j) = 1$ for each p_j such that $F_2(p_j) = 1$. If $F_1 = F_2$, then $F_1 \geqslant F_2$ and $F_2 \geqslant F_1$. A function F_1 *properly contains* a function F_2 written $F_1 > F_2$, if F_1 contains F_2 and if F_2 does not contain F_1. Switching functions F_1 and F_2 are called *comparable* if $F_1 \geq F_2$ or if $F_2 \geq F_1$.

If in the expression for any switching function F all variables in a set S_1 are set equal to unity and all variables in a disjoint set S_2 are set equal to zero, a new function is formed which is called a *reduced function* and which is denoted by $F(S_1 = 1, S_2 = 0)$. Letting the union of S_1 and S_2 be denoted by S and letting s_1 and s_2 denote the sums of the coefficients of the variables of S_1 and S_2, respectively, the following four properties[4] of reduced functions and their functional containment relations are presented without proof. Derivations of these are found in (2).

PROPERTY 6.[5] *If* F *is linearly separable with separating plane equation* $f = T$, *then* $F(S_1 = 1, S_2 = 0)$ *is linearly separable with separating plane equation* $f(S_1 = 1, S_2 = 0) = T$; *or, what is the same thing,* $f(S = 0) = T - s_1$. *Moreover, all reduced functions* $F(S_p = 1, S_q = 0)$ *such that the union of* S_p *and* S_q *is* S *have the same separating plane function.*

PROPERTY 7. *All linearly separable functions which can have the same separating plane function are comparable.*

[3] A more detailed discussion is given in (2).

[4] Properties 6, 7, and 8 are similar to results derived by Paull and McCluskey (8).

[5] The properties in this paper are not numbered consecutively, inasmuch as corresponding properties here and in (2) are given the same number.

PROPERTY 8. *If* F *is a linearly separable function then* $F(S_1 = 1, S_2 = 0) > F(S_1 = 0, S_2 = 1)$ *implies that* $s_1 > s_2$ *in all separating plane functions for* F. *Furthermore, if the sets* S_3 *and* S_4 *are such that the union of* S_3 *and* S_4 *equals the union of* S_1 *and* S_2, *then* $F(S_1 = 1, S_2 = 0) > F(S_3 = 1, S_4 = 0)$ *implies that* $s_1 > s_3$ *in all separating plane functions for* F.

If $F(S_1 = 1, S_2 = 0) = F(S_1 = 0, S_2 = 1)$, nothing in general can be said about the comparability of s_1 and s_2. However, if the sets S_1 and S_2 are single variables, say x_i and x_k, then the following property is applicable.

PROPERTY 9. *If* F *is a linearly separable function then* $F(x_i = 1, x_k = 0) = F(x_i = 0, x_k = 1)$ *implies that* F *is symmetric in* x_i *and* x_k. *Furthermore, if* F *is symmetric in* m *variables, then for every separating plane equation in which the coefficients of these* m *variables are not equal, there is another separating plane equation identical to the first except that these* m *coefficients are all equal.*

Property 8 shows how the containment properties of reduced functions place restrictions on the sums of coefficients s_1 and s_2. Property 10 gives additional restriction on s_1 and s_2 when a separating plane function for the reduced functions is specified.

PROPERTY 10. *If* F *is a linearly separable function and if* $F(S_1 = 1, S_2 = 0)$ *and* $F(S_3 = 1, S_4 = 0)$, *where the union of* S_1 *and* S_2 *equals the union of* S_3 *and* S_4 *equals* S, *have separating plane function* f′ *with gaps* $u_1{:}l_1$ *and* $u_3 - l_3$, *respectively, then the coefficientco mbination* (*linear combination of coefficients*) $s_1 - s_3$ *is constrained by* $u_3 - l_1 > s_1 - s_3 > l_3 - u_1$, *in any separating plane function* f *for* F *such that* $f' = f(S = 0)$.

Since F is linearly separable with separating plane equation $f = T$, then by Property 6, $F(S_1 = 1, S_2 = 0)$ and $F(S_3 = 1, S_4 = 0)$ have separating plane equations $f' = T_1 = T - s_1$ and $f' = T_3 = T - s_3$ where T_1 and T_3 lie inside their respective gaps $G_1 = u_1{:}l_1$ and $G_3 = u_3{:}l_3$. It follows, therefore, that $T_1 + s_1 = T_3 + s_3$ and that $s_1 - s_3 = T_3 - T_1$. Moreover, since $u_1 \geqslant T_1 > l_1$ and $u_3 \geqslant T_3 > l_3$, we have $s_1 - s_3$ constrained according to $u_3 - l_1 > s_1 - s_3 > l_3 - u_1$. This proves Property 10.

Thus far we have considered general properties of reduced functions. We now consider some of the implications of these results in special cases. The first of these is concerned with the conditions which[6] $F = x_k F_1 + F_0$ must satisfy so that F_1 and F_0 are linearly separable functions when F is linearly separable.

[6] In this paper, + denotes Boolean addition.

PROPERTY 11. *Let* F *be a switching function factored so that* $F = x_k F_1 + F_0$ *where* x_k *does not appear in the min-term expressions for* F_1 *or* F_0. *If* F *is linearly separable and* $F_1 \geqslant F_0$, *then* F_1 *and* F_0 *are reduced functions of* F; *hence they are linearly separable. Moreover, if* $a_k x_k + f'$ *is a separating plane function for* F, *then* f′ *is a separating plane function for* F_1 *and* F_0.

The validity of Property 11 follows immediately from Property 6 since $F(x_k = 0) = F_0$ and $F(x_k = 1) = F_1 + F_0 = F_1$ when $F_1 \geqslant F_0$.

A question related to that considered in Property 11 concerns the requirements on functions F_1 and F_0 in order that $x_k F_1 + F_0$ be linearly separable. These results are given by Properties 12, 13 and 14, the proofs of which follow Property 14.

PROPERTY 12. *If* F_1 *and* F_0 *are linearly separable functions in whose min-term expressions* x_k *does not appear and if* F_1 *and* F_0 *can have the same separating plane function, then* $x_k F_1 + F_0$ *is a linearly separable function.*

Let f' be any separating plane function for both F_1 and F_0 and let the gap of the corresponding maps be $u_1 : l_1$ and $u_0 : l_0$, respectively. To consider the relation between f' and the separating plane functions for $F = x_k F_1 + F_0$, let $L(a_k) = l_0 - u_1$ and $U(a_k) = u_0 - l_1$.

The a_k-*range* defined by F_1 and F_0 is the set of all real numbers a_k such that $L(a_k) < a_k < U(a_k)$. $L(a_k)$ is called the *lower bound* of the range and $U(a_k)$ is called the *upper bound*. We represent the range by $R(a_k)$ or $U(a_k) \star L(a_k)$ and we often write $R(a_k) = U(a_k) \star L(a_k)$.

PROPERTY 13. *Let* f′ *be any separating plane function for both* F_1 *and* F_0 *where* $F_1 > F_0$. *Then* $a_k x_k + f'$ *is a separating plane function for* $F = x_k F_1 + F_0$ *if and only if the value of* a_k *lies in* $R(a_k)$.

The proof of Property 13 provides a useful relationship between the maps of F_1 and F_0 due to separating plane function f' and the map of $F = x_k F_1 + F_0$ due to separating plane function $f = a_k x_k + f'$ when $F_1 > F_0$. This relationship is described by Property 14 and is illustrated by Fig. 3. For this description we define a *displaced map*. We say that the map $\{f(p_j) + a_k, F(p_j)\}$ is the map $\{f(p_j), F(p_j)\}$ displaced by a_k. In the visualization, each point of the original map is increased by the amount a_k.

PROPERTY 14.

(*a*) *When* $F_1 > F_0$ *the map of* $F = x_k F_1 + F_0$ *due to* $a_k x_k + f'$ *is the union of the map of* F_0 *due to* f′ *and the map of* F_1 *due to* f′ *displaced by* a_k. *This implies that the gap of the map of* F *is the inter-*

section of the gap of the map of F_0 *and the gap of the displaced map of* F_1. *If the gaps of* F_0 *and* F_1 *are denoted by* $u_0:l_0$ *and* $u_1:l_1$, *respectively, then the gap of* F, $u:l$, *is given by*

$$u:l = Min \left\{ \begin{matrix} u_1 + a_k \\ u_0 \end{matrix} \right\} : Max \left\{ \begin{matrix} l_1 + a_k \\ l_0 \end{matrix} \right\}.$$

(*b*) *The gap length* g *of the map of* F *is the following function of* a_k *for* a_k *in* $R(a_k)$

$$g = Min \left\{ \begin{matrix} g_m \\ U(a_k) - a_k \\ a_k - L(a_k) \end{matrix} \right\}$$

where g_m *is the smaller of the gap lengths of* F_1 *and* F_0.

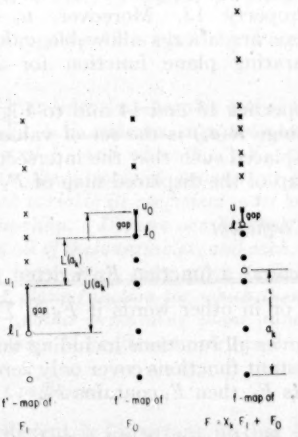

FIG. 3. Maps of F_1, F_0 and $F = x_kF_1 + F_0$.

We now sketch the proofs of Properties 12, 13 and 14. When $F_0 \geq F_1$, the validity of Property 12 is evident since $x_kF_1 + F_0 = F_0$. When $F_1 > F_0$, Property 12 is implied by Properties 13 and 14. We consider, therefore, the proofs of Properties 13 and 14.

The points for which $F = x_kF_1 + F_0$ are unity are either points for which $x_k = 1$ and $F_1 = 1$ or they are points for which $x_k = 0$ and $F_0 = 1$. Now consider the map of F due to $f = a_kx_k + f'$. For all points for which $x_k = 0$, this map of F is clearly identical to the map of F_0 due to f'. For all points for which $x_k = 1$, $f(p_j) = a_k + f'$, and thus for

these points the map of F due to f is the map of F_1 due to f' and displaced by a_k. It follows, therefore, that the map of F due to f is the union of the maps of F_0 due to f' and of F_1 due to f' and displaced by a_k (see Fig. 3). Moreover, since u is the smallest $f(p_j)$ for which F equals unity and l is the largest $f(p_j)$ for which F equals zero it follows that

$$u = \text{Min}\left\{\begin{matrix} u_0 \\ u_1 + a_k \end{matrix}\right\} \quad \text{and} \quad l = \text{Max}\left\{\begin{matrix} l_0 \\ l_1 + a_k \end{matrix}\right\}$$

and that the gap $u:l$ is the intersection of the gap of F_0 and the gap of F_1 displaced by a_k, thus proving Property 14*a*. Property 14*b* follows from a simple observation of these results.

It is clear that f is a separating plane function for F if and only if the gap of F_0 has a non-void intersection with the gap of F_1 displaced by a_k and that these gaps will intersect if and only if $u_0 - l_1 > a_k > l_0 - u_1$. This establishes Property 13. Moreover, $u_0 - l_1 > l_0 - u_1$ since $F_1 > F_0$; hence, there are always allowable values for a_k such that $a_k x_k + f'$ is a separating plane function for F. This establishes Property 12.

Reference to Properties 13 and 14 and to Fig. 3 shows that when $F_1 > F_0$, then the range $R(a_k)$ is the set of values of a_k by which the map of F_1 can be displaced such that the intersection of the gap of the map of F_0 and the gap of the displaced map of F_1 is non-void.

b. The Largest Coefficient

A function F_1 "*covers*" a function F_0, written $F_1 \supset F_0$, if for every x_k, $F_1 \geqslant F_0(x_k = 1)$ or in other words if $F_1 \geqslant \sum_{k=1}^{n} \cdot F_0(x_k = 1)$.[7] Unit constant functions cover all functions including unit and zero constant functions. Zero constant functions cover only zero constant functions. Note that if F_1 covers F_0, then F_1 contains F_0.

EXAMPLE 1. If $F_1 = x_1 + x_2x_3$ and $F_0 = x_1x_2x_3$, then F_1 covers F_0 since

$$F_1 = x_1 + x_2x_3 \geqslant F_0(x_1 = 1) + F_0(x_2 = 1) + F_0(x_3 = 1)$$
$$= x_2x_3 + x_1x_3 + x_1x_2.$$

If F is a linearly separable switching function, then the identity of the largest coefficient in the associated separating plane function is detectable from the covering and containment properties of F. The relations whereby this is possible are given by Properties 15 and 16, the proofs of which are found in (2).

[7] $\Sigma\cdot$ denotes Boolean addition.

PROPERTY 15. *If* F *is linearly separable, and* x_h *is the variable with the largest coefficient in the associated separating plane function, then* F *can be factored in the form* $F = x_hF_1 + F_0$, *where* x_h *does not appear in the min-term expressions for* F_1 *and* F_0 *and where* $F_1 \supset F_0$. *Moreover,* F_1 *and* F_0 *are linearly separable and can have the same separating plane function.*

EXAMPLE 2. In the linearly separable function

$$F = x_1(x_2 + x_3 + x_4x_5 + x_4x_6 + x_5x_6) + (x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5x_6),$$

the variable x_1 can have the largest coefficient in the associated separating plane function because

$$(x_2 + x_3 + x_4x_5 + x_4x_6 + x_5x_6) \supset (x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5x_6).$$

The converse of Property 15 is given by Property 16 the proof of which is given in (2).

PROPERTY 16. *If* F *is linearly separable and if there is any variable* x_k *such that* $F = x_kF_1 + F_0$, *where* x_k *does not appear in the min-term expressions for* F_1 *and* F_0 *and where* $F_1 \supset F_0$, *then* x_k *can have the largest coefficient in the associated separating plane function. If there is only one such variable its coefficient must be the largest in every separating plane function. If there are* M *such variables, the function is symmetric in all of these variables, and each of the corresponding coefficients must be larger than every other coefficient. Furthermore, for every separating plane function for which these* M *coefficients are not equal, there is another separating plane function identical with the first except that these* M *coefficients are equal.*

c. The Smallest Coefficient

The smallest coefficient is important in that its value is an upper bound on gap length. The relations between these quantities are given by Properties 17 and 18, the proofs for which are found in (2).

PROPERTY 17. *If the gap length of a map is denoted by* g *and if* a_s *is the smallest coefficient of the separating plane function, then* $g \leqslant a_s$.

Another bound on gap size, one that can be determined without actually knowing either the size or the ordering of the coefficients, follows easily from Property 17.

PROPERTY 18. *If* M *is the maximum number of variables in any min-term of* F *and if* g/T *is the relative gap for the map of* F, *then*

$$\frac{g}{T} \leqslant \frac{1}{M - \frac{1}{2}}.^{8}$$

Moreover, if n *is the number of variables in* F, *then for all* n *and for all* $M \leqslant n$, *there is a linearly separable switching function which has*

$$\frac{g}{T} = \frac{1}{M - \frac{1}{2}}.^{9}$$

D. Coefficient Ordering

a. Necessary Properties

This section describes a procedure for determining the order which is required of the coefficients of a separating plane function for a given linearly separable function. Properties 8 and 9 suggest one such procedure; however, it would require the formation of a number of reduced functions and the testing of their comparability.

The procedure to be presented here is equivalent; however, it is simpler since it depends only upon the number of times certain variables appear in certain min-terms. The basis for the procedure is Property 21, the derivation of which is given in (2).

PROPERTY 21. *Let* x_i *and* x_k *be any two variables of the switching function* F. *If*

$$F(x_i = 1, x_k = 0) > F(x_i = 0, x_k = 1),$$

then there must be some integer t, $M \geq t \geq m$ (*where* m *is the smallest and* M *the largest number of variables in any min-term*) *such that in the group of min-terms having exactly* t *variables,* x_i *appears more times than* x_k. *Moreover, if* $t > m$ *then in those min-terms having fewer than* t *variables,* x_i *appears the same number of times as* x_k.

b. Ordering Procedure

The procedure is essentially a mechanization of Property 21. For linearly separable functions it gives the same ordering as would be given by checking comparability of reduced functions. The procedure will be described in sequential format and illustrated by Example 3.

(*i*) Count the number of times each variable appears in those min-terms of *F* which contain exactly *m* variables where *m* is the minimum number of variables in any min-term.

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[8] The inequality is actually valid only for $\frac{g}{T} \leqslant 2$. However, if all of the coefficients are positive and if *T* is in the center of the gap, then g/T will always be less than or equal to 2.

[9] A class of functions for which this is true is the *M* out of *n* symmetrical functions.

This procedure divides the variables into disjoint *groups* where all variables in a given group appear the same number of times, and where one group contains those variables that appear no times. According to Property 20 the coefficients are now partially ordered in that all of those variables in the group of variables appearing n_i times have larger coefficients than those appearing n_j times if $n_i > n_j$.

(*ii*) Consider those min-terms of F that contain exactly $m + 1$ variables. Count the number of appearances in these terms of those variables which belong to groups that contain more than one variable.

This procedure divides each group of variables into subgroups, where all the variables in a given subgroup appear the same number of times. The variables in each subgroup are partially ordered in that all those variables appearing n_i times (in the counting of step (*ii*)) have larger coefficients than those appearing n_j times if $n_i > n_j$. Variables that were ordered by step (*i*) retain their ordering.

(*iii*) Repeat the process until the variables are all ordered or until all terms of the function have been used.

In the latter case the function is symmetric with respect to the variables of each remining group.

The test and realization procedure for linearly separable switching functions, which is described in Section III, requires a procedure that identifies only the variables that can have the largest coefficient in the corresponding separating plane function when the switching function is linearly separable. This variable will be identified after step (*i*) of the Ordering Procedure if exactly one of the variables occurs n_1 times where $n_1 > n_i$ for all $i \neq 1$. If the n_1 group contains more than one variable, then only these variables need be considered by step (*ii*). If step (*ii*) does not yield the variable which can have the largest coefficient, then only those occurring the largest number of times in step (*ii*) need be considered in step (*iii*) and so on.

EXAMPLE 3. From Example 2

$$F = x_1x_2 + x_1x_3 + x_1x_4x_5 + x_1x_4x_6 + x_1x_5x_6 + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5x_6.$$

Its coefficients are ordered as follows:

i. The smallest min-terms are those containing two variables

x_1x_2 and x_1x_3.

Since x_1 appears the most times in these min-terms, a_1 is the largest coefficient. One of the coefficients a_2 or a_3 is the second and one the third largest coefficient. This information is given by the larger min-terms.

ii. The min-terms containing three variables are

$$x_1x_4x_5, \quad x_1x_4x_6, \quad x_1x_5x_6, \quad x_2x_3x_4, \quad x_2x_3x_5 \quad \text{and} \quad x_2x_4x_5.$$

In these terms x_2 appears three times and x_3 twice; therefore, a_2 is the second and a_3 the third largest coefficient. Of the other variables, x_4 appears four times, x_5 four times and x_6 twice. Since the only four-variable min-term contains x_3, x_4, x_5 and x_6 symmetrically, a_4 and a_5 can be equal and a_6 is smaller than both a_4 and a_5.

iii. Thus the coefficients are ordered

$$a_1 > a_2 > a_3 > a_4 = a_5 > a_6.$$

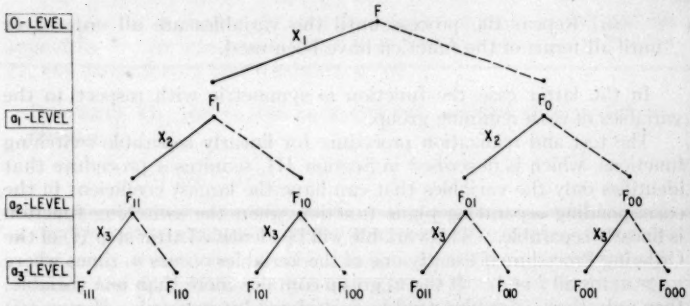


FIG. 4. Format of the function tree.

III. TESTS AND REALIZATION

This section is concerned with the development of a procedure for testing whether or not a given switching function is linearly separable and for realizing all of those which belong to the linearly separable class. The test consists of two parts: *Decomposition* and *Reconstruction*. It begins with Decomposition.

A. Decomposition

a. Description of Decomposition

The decomposition procedure assumes that the given switching function F is expressed in min-term form and that it is unate with only

positive variables or that it has been transformed to an "equivalent" positive variable function using Property 4 of Section II. The changes in the procedure which are necessary when the given function is expressed in truth-table form are given in Section IV-*B* of (2).

Decomposition produces a tree of reduced functions called the *function tree.* The format of this tree is given by Fig. 4; the procedure itself is illustrated by Example 4.

(*i*) Using the Ordering Procedure of Section II, *D-b*, determine a variable of F which can have the largest coefficient in the separating plane function if F is linearly separable. Denote this variable as x_1 and factor F into $F = x_1F_1 + F_0$, where x_1 does not appear in the min-term expressions for F_1 and F_0. F is the function on the 0-level of the function tree; F_1 and F_0 are the functions on the a_1-level. Between F and F_1 the tree has a solid line, between F and F_0 the line is dotted.

If F is linearly separable, then by Properties 11 and 15, F_1 and F_0 are the reduced linearly separable functions, $F(x_1 = 1)$ and $F(x_1 = 0)$, respectively, and $F_1 \supset F_0$.[10] If F has separating plane function $a_1x_1 + f_1$, then f_1 is a separating plane function of both F_1 and F_0. It is necessary, therefore, that F_1 and F_0 have at least one variable in common which can have the largest coefficient in the separating plane function of each.

The next step of the procedure identifies this variable if it exists. Constant functions are not considered further since they are linearly separable and any variable can have the largest coefficient in their separating plane function. To continue the procedure, let $k = 1$.

(*ii*) The set of all possible functions on the a_k-level of the function tree is $\{F_{\delta_1\delta_2\cdots\delta_k}\}$ where $\delta_1, \delta_2, \cdots, \delta_k = 0, 1$. Some of these may be constant functions. Using the Ordering Procedure of Section II, *D-b*, determine for each non-constant function on the a_k-level, the group of variables which can have the largest coefficient in its separating plane function. Select a variable which is common to all groups and denote it by x_{k+1}. If such a variable does not exist then the original function is not linearly separable. Otherwise, factor each non-constant function as

$$F_{\delta_1\delta_2\cdots\delta_k} = x_{k+1}F_{\delta_1\delta_2\cdots\delta_k1} + F_{\delta_1\delta_2\cdots\delta_k0}$$

where x_{k+1} does not appear in the min-term expressions for $F_{\delta_1\delta_2\cdots\delta_k1}$ and $F_{\delta_1\delta_2\cdots\delta_k0}$. This set $\{F_{\delta_1\delta_2\cdots\delta_{k+1}}\}$ consists of the functions on the

[10] It is possible to determine from F_1 and F_0 whether or not $F_1 \supset F_0$. If this condition is not satisfied, then it is known that F is not linearly separable. It is unnecessary to check this condition, however, and it is probably uneconomic to do so except for functions with relatively few terms in the min-term expression.

a_{k+1}-level of the function tree. Between each $F_{\delta_1\delta_2\cdots\delta_k}$ and $F_{\delta_1\delta_2\cdots\delta_k 1}$ there is a solid line; between each $F_{\delta_1\delta_k\cdots\delta_2}$ and $F_{\delta_1\delta_2\cdots\delta_k 0}$ the line is dotted.

Although constant functions were explicitly excluded from further consideration by (*ii*) this was a convenience rather than a necessity. If they are not omitted, the procedure can be carried out in the normal fashion; it will be found that each constant function, 0 or 1, connects, respectively, to only 0 or 1 constant function on the lower levels and that in general the constant functions have the same properties as the other functions in the tree. It is convenient, therefore, to omit the constant functions; if needed, they can easily be reinserted. Function trees to which these constant functions have been completely or partially supplied are called *complete* or *partially complete* function trees. In the main, the properties of a function tree are independent of whether or not it is complete. We make no distinction between the two when considering the properties of the function tree and we refer to both as the function tree unless the distinction is explicitly expressed.

If each $F_{\delta_1\delta_2\cdots\delta_k}$ is linearly separable, then by Properties 11 and 15, $F_{\delta_1\delta_2\cdots\delta_k 1}$ and $F_{\delta_1\delta_2\cdots\delta_k 0}$ are the reduced linearly separable functions $F_{\delta_1\delta_2\cdots\delta_k}(x_{k+1} = 1)$ and $F_{\delta_1\delta_2\cdots\delta_k}(x_{k+1} = 0)$, respectively, and $F_{\delta_1\delta_2\cdots\delta_k 1} \supset F_{\delta_1\delta_2\cdots\delta_k 0}$.[11] If each $F_{\delta_1\delta_2\cdots\delta_k}$ can have the same separating plane function, say $a_{k+1}x_{k+1} + f_{k+1}$, then each $F_{\delta_1\delta_2\cdots\delta_{k+1}}$ can have separating plane function f_{k+1}, hence all a_{k+1}-level functions can have the same separating plane function. It is necessary, therefore, that the set of a_{k+1}-level functions have at least one variable in common which can have the largest coefficient in the separating plane function of each. The next step of the procedure identifies this variable if it exists.

(*iii*) If $k < n - 2$, add 1 to k and repeat the procedure starting with (*ii*). If $k = n - 2$, the function tree is completed and the a_{k+1}-level functions are either constant functions or are the function x_n.

Since x_n is a linearly separable function, it follows immediately that all a_{n-1}-level functions are linearly separable and can have the same separating plane function.

EXAMPLE 4. The function tree for the switching function of Example 2 is given in Fig. 5.

[11] It is possible to determine from each $F_{\delta_1\delta_2\cdots\delta_k 1}$ and each $F_{\delta_1\delta_2\cdots\delta_k 0}$ whether or not $F_{\delta_1\delta_2\cdots\delta_k 1} \supset F_{\delta_1\delta_2\cdots\delta_k 0}$. If this condition is not satisfied, then it is known that $F_{\delta_1\delta_2\cdots\delta_k}$ is not linearly separable and this implies that F is not a linearly separable function. It is unnecessary to check this condition, however, and it is probably uneconomic to do so except for functions with relatively few terms in the min-term expression.

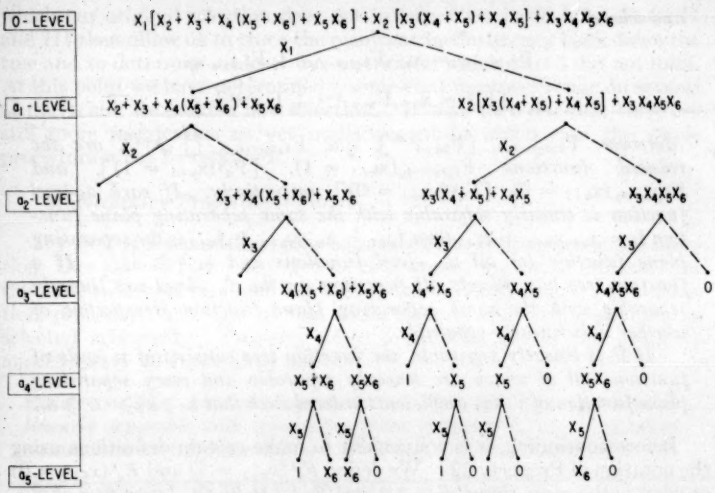


FIG. 5. Function tree for Example 4.

b. Discussion of the Function Tree

The function tree for a completely decomposed switching function consists of n-levels which are denoted by 0, a_1, a_2, $\cdots$, a_{n-1}. The function on the 0-level is the original switching function F. The functions on each a_k-level are reduced functions of F. They are denoted by $F_{\delta_1\delta_2\cdots\delta_k}$ where $\delta_1, \delta_2, \cdots, \delta_k = 0, 1$. At times it is more convenient to use a decimal subscript notation to represent the functions of the tree. With this notation the a_k-level function $F_{\delta_1\delta_2\cdots\delta_k}$ is denoted by $F_q{}^k$ where q is the decimal equivalent of the binary subscript. We use both notations in Property 22 to express the relations between the functions on any a_k-level of the tree and those on the a_{k+1}-level. The decimal subscript notation is enclosed in square brackets. The validity of Property 22 follows immediately from the description of Decomposition.

PROPERTY 22. *To each* a_k*-level non-constant function* $F_{\delta_1\delta_2\cdots\delta_k}$, $[F_q{}^k]$, *there is connected by a solid line the* a_{k+1}*-level function* $F_{\delta_1\delta_2\cdots\delta_k 1}$, $[F_{2q+1}{}^{k+1}]$, *and by a dotted line the* a_{k+1}*-level function* $F_{\delta_1\delta_2\cdots\delta_k 0}$, $[F_{2q}{}^{k+1}]$, *where*

$$F_{\delta_1\delta_2\cdots\delta_k 1} \supset F_{\delta_1\delta_2\cdots\delta_k 0}[F_{2q+1}{}^{k+1} \supset F_{2q}{}^{k+1}]$$

and where

$$F_{\delta_1\delta_2\cdots\delta_k} = x_{k+1}F_{\delta_1\delta_2\cdots\delta_k 1} + F_{\delta_1\delta_2\cdots\delta_k 0}$$

$$[F_q{}^k = x_{k+1}F_{2q+1}{}^{k+1} + F_{2q}{}^{k+1}].$$

Moreover, $F_{\delta_1\delta_2\cdots\delta_k 1}$, $[F_{2q+1}{}^{k+1}]$, *and* $F_{\delta_1\delta_2\cdots\delta_k 0}$, $[F_{2q}{}^{k+1}]$, *are the reduced functions* $F_{\delta_1\delta_2\cdots\delta_k}(x_{k+1} = 1)$, $[F_q{}^k(x_{k+1} = 1)]$, *and* $F_{\delta_1\delta_2\cdots\delta_k}(x_{k+1} = 0)$, $[F_q{}^k(x_{k+1} = 0)]$, *respectively. If each* a_k*-level function is linearly separable with the same separating plane function* $f_k = a_{k+1}x_{k+1} + f_{k+1}$ *then* $f_{k+1} = a_{k+2}x_{k+2} + f_{k+2}$ *is the separating plane function for all* a_{k+1}*-level functions and* $a_{k+1} \geqslant a_{k+2}$. *If a function tree has* n*-levels all functions on the* a_{n-1}*-level are linearly separable with the same separating plane function irrespective of whether* F *is linearly separable.*

If F *is linearly separable, the function tree consists of* n *levels of functions all of which are linearly separable and every separating plane function of* F *has coefficients ordered such that* $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$.

Before continuing, it is convenient to make certain definitions using the notation of Property 22. We define $F_q{}^k(x_{k+1} = 1)$ and $F_q{}^k(x_{k+1} = 0)$ or what is the same thing, $F_{2q+1}{}^{k+1}$ and $F_{2q}{}^{k+1}$ to be the *lower pair* of $F_q{}^k$. We sometimes call such sets of two functions an a_{k+1}-level *function pair* and call $F_q{}^k$ the *upper function* of the pair.

Although Decomposition yields a function tree of n-levels for all linearly separable functions, there are also non-linearly separating functions for which n-level trees are obtained. An example is the six-variable function of Example 7. We see, therefore, that the realization of a function tree is a necessary but not sufficient test for a switching function to be linearly separable. Sufficiency is provided by Reconstruction.

B. Reconstruction

The purpose of Reconstruction is to determine whether or not a function for which an n-level function tree has been obtained is linearly separable and if it is, to provide a realization. Reconstruction is divided into three parts in order to simplify the description. Part I is an elementary procedure for determining the coefficients of a separating plane function on the assumption that the switching function is linearly separable. Parts II and III are concerned with those cases when the procedure of Part I cannot be continued.

The complete test is somewhat iterative in nature. In Reconstruction Part I we check certain simple range constraints, determine a range on each a_k and make a selection within that range. If it turns out that some range constraint not checked in Part I places restrictions on a_k

which our original selection does not satisfy, Part I will fail. Parts II and III then allow us to trace the resultant inconsistency back down the tree and to determine at least one restriction which Part I did not fulfil. At this point we have determined a somewhat narrower range on several of the a_k and we make a new selection. It may turn out that there are still more restrictions as yet undiscovered in which case the whole procedure must be repeated.

a. Necessary Properties—Part I

If $f = T$ is a separating plane equation for a linearly separable function, then $kf = kT$, for any $k > 0$, is a separating plane equation for the same switching function. It follows immediately that in any realization procedure any one of the coefficients or the threshold can be selected arbitrarily. Property 24 is a combination of this property and Property 22 as we shall use it.

PROPERTY 24. *All functions on the* a_{n-1}*-level of a function tree are linearly separable with separating plane function* $f_{n-1} = a_n x_n$ *where* a_n *can have any positive value. Moreover, if* F *is linearly separable, then it has separating plane function* $F = \sum_{k=1}^{n-1} a_k x_k + f_{n-1}$ *for every* f_{n-1}.

Consider any a_k-level of the tree on which all switching functions are linearly separable and have separating plane function f_k. By Property 22 each a_k-level function pair, $F_{2q+1}{}^k$ and $F_{2q}{}^k$, combines to form the a_{k-1}-level function $F_q{}^{k-1} = x_k F_{2q+1}{}^k + F_{2q}{}^k$. Moreover, $F_{2q+1}{}^k \supset F_{2q}{}^k$ and therefore by Property 13 each $F_q{}^{k-1}$ is linearly separable and has separating plane function $f_{k-1} = a_k x_k + f_k$ if and only if a_k is selected in the range $R_q(a_k)$ defined by the lower function pair of $F_q{}^{k-1}$ (by $F_{2q+1}{}^k$ and $F_{2q}{}^k$).

Thus each a_k-level function pair defines a range $R_q(a_k) = U_q(a_k) \star L_q(a_k)$. The *common* a_k*-range* is denoted by $\overline{R(a_k)} = \overline{U(a_k)} \star \overline{L(a_k)}$, where $\overline{U(a_k)} = \mathrm{Min}_q\{U_q(a_k)\}$ and $\overline{L(a_k)} = \mathrm{Max}_q\{L_q(a_k)\}$. We refer to $\overline{R(a_k)}$ as the intersection of all $R_q(a_k)$. We say that $\overline{R(a_k)}$ is void or does not exist, if $\overline{U(a_k)} \leqslant \overline{L(a_k)}$, and we denote this by $\overline{R(a_k)} = \phi$. We let $|\overline{R(a_k)}|$ denote the *length of* $\overline{R(a_k)}$ which for non-void ranges we define as $|\overline{R(a_k)}| = \overline{U(a_k)} - \overline{L(a_k)}$.

Property 25 is an immediate consequence of this definition.

PROPERTY 25. *Let* F *be a linearly separable function and let* $f_k = \sum_{q=k+1}^{n} a_q x_q$ *be a separating plane function for all switching functions on the* a_k*-level of the function tree. Then all* a_{k-1}*-level functions*

are linearly separable and can have for a separating plane function $f_{k-1} = a_k x_k + f_k$, *if an only if* $\overline{R(a_k)} \neq \phi$ *and* a_k *is selected in* $\overline{R(a_k)}$.

Let all a_k-level functions be linearly separable with separating plane function f_k. We define the a_k-level *common gap*, denoted by $\overline{g_k}$, to be the smallest of all a_k-level gap lengths as defined by f_k. By a straightforward argument, the following property is established from Property 14.

PROPERTY 26. *Let all* a_k*-level switching functions have separating plane function* f_k *and let* $f_{k-1} = a_k x_k + f_k$ *be the separating plane function for all* a_{k-1}*-level switching functions:*

If $|\overline{R(a_k)}| \geqslant 2\overline{g_k}$, *then* $\overline{g_{k-1}}$ *is a maximum and equal to* $\overline{g_k}$ *for all values of* a_k *such that:*

$$\overline{L(a_k)} + \overline{g_k} \leqslant a_k \leqslant \overline{U(a_k)} - \overline{g_k}.$$

If $|\overline{R(a_k)}| < 2\overline{g_k}$, *then* $\overline{g_{k-1}}$ *is a maximum and equal to* $\frac{1}{2}|\overline{R(a_k)}|$ *for a value of* a_k *such that*

$$a_k = \tfrac{1}{2}[\overline{U(a_k)} + \overline{L(a_k)}].$$

b. Description of Reconstruction—Part I

Reconstruction—Part I is a procedure for determining a separating plane function for a switching function which has a function tree. It may or may not yield a realization. If it does not, that is, if it is impossible to complete the procedure, either the given function is not linearly separable or certain conditions which are important to the realization have not been satisfied. This is entirely possible since Part I utilizes only a small part of the information which is available in the tree structure. In either case, however, Part I then becomes the first part of a more general reconstruction procedure.

Reconstruction—Part I is performed directly on the function tree obtained by Decomposition (not on the complete tree) and it begins on the a_{n-1}-level. The procedure is described in sequential format and is illustrated by Example 5.

According to Property 22, all a_{n-1}-level functions are linearly separable and can have the same separating plane function $f_{n-1} = a_n x_n$. According to Property 24 the value of a_n is arbitrary therefore,

(*i*) Choose $a_n = 1$.

Each a_{n-1}-level function F_q^{n-1} has separating plane function $f_{n-1} = x_n$. The map of each due to f_{n-1} has exactly two points and the gap of each is as follows: $F_q^{n-1} = x_n$ has gap 1:0, $F_q^{n-1} \equiv 1$ has gap $0:-\infty$, and $F_q^{n-1} \equiv 0$ has gap $\infty:1$.

(*ii*) Below each a_{n-1}-level function indicate the appropriate gap. Note that $\overline{g_{n-1}} = 1$. Continue the procedure by letting $k = n - 1$.

Since all a_k-level functions are linearly separable with separating plane function f_k, then according to Property 25 all functions on the a_{k-1}-level are linearly separable and all can have separating plane function $f_{k-1} = a_k x_k + f_k$ if and only if $\overline{R(a_k)} \neq \phi$ and a_k belongs to $\overline{R(a_k)}$.

(*iii*) For each function pair $F_{2q+1}{}^k$, $F_{2q}{}^k$ determine the corresponding $R_q(a_k) = U_q(a_k) \star L_q(a_k)$ where $U_q(a_k) = u_{2q}{}^k - l_{2q+1}{}^k$ and $L_q(a_k) = l_{2q}{}^k - u_{2q+1}{}^k$ and from these determine $\overline{R(a_k)} = \overline{U(a_k)} \star \overline{L(a_k)}$ where $\overline{U(a_k)} = \text{Min}_q\{U_q(a_k)\}$ and $\overline{L(a_k)} = \text{Max}_q\{L_q(a_k)\}$.

(*iv*) If $\overline{R(a_k)} = \phi$, that is, if $\overline{U(a_k)} \leqslant \overline{L(a_k)}$, then it is necessary to use Part II to continue Reconstruction. For this we need the set of $R_q(a_k)$ as determined by step (iii). We denote this set by $\{R(a_k)\}$. If $\overline{R(a_k)} \neq \phi$, select the value of a_k according to

$$a_k = \text{Min}\left\{\frac{\overline{L(a_k)} + \overline{g_k}}{\frac{1}{2}[\overline{U(a_k)} + \overline{L(a_k)}]}\right\}.$$

According to Property 26 this is the smallest value of a_k such that $\overline{g_{k-1}}$ is a maximum when the separating plane function is $f_{k-1} = a_k x_k + f_k$.

(*v*) Determine the gap for each function on the a_{k-1}-level as follows: If $F_q{}^{k-1} \equiv 1$, its gap is $0 : -\infty$. If $F_q{}^{k-1} \equiv 0$, its gap is $\infty : \sum_{n=k}^{k} a_i$. If $F_q{}^{k-1}$ is not a constant function, then its gap is $u_q{}^{k-1} : l_q{}^{k-1}$ where

$$u_q{}^{k-1} = \text{Min}\left\{\frac{u_{2q}{}^k}{u_{2q+1}{}^k + a_k}\right\} \qquad \text{and} \qquad l_q{}^{k-1} = \text{Max}\left\{\frac{l_{2q}{}^k}{l_{2q+1}{}^k + a_k}\right\}.$$

The a_{k-1}-level common gap is the smallest of these gap lengths and is given by

$$\overline{g_{k-1}} = \text{Min}_q\{u_q{}^{k-1} - l_q{}^{k-1}\}.$$

The validity of step (*v*) for constant functions follows from the definition of gap and for non-constant functions follows from Property 14.

(*vi*) If $k > 1$ subtract 1 from k and repeat the procedure beginning with the paragraph immediately following (*ii*).

If $k = 1$ then there is only one a_{k-1}-level function and it is the original switching function. It is linearly separable with separating

plane function $f_{k-1} \equiv f$, and the gap length of its map is g_{k-1}. The threshold T is selected as the midpoint of the gap.

All switching functions that can be reconstructed from the function tree by Part I are linearly separable and have a known separating plane function. Note that the constant functions which were omitted would only have produced ranges of the form $\infty \star -\infty$ and hence would not have affected the final result.

It is clear from step (*iv*), however, that on some a_k-level it may be impossible to continue reconstruction by Part I because $\overline{R(a_k)} = \phi$. This must occur for all functions which are not linearly separable, but it may also occur for some linearly separable functions. The reasons for this are the basis for Reconstruction—Part II and are the subject of the next section.

EXAMPLE 5. The application of Reconstruction—Part I to the function tree of Example 4 is shown in Fig. 6.

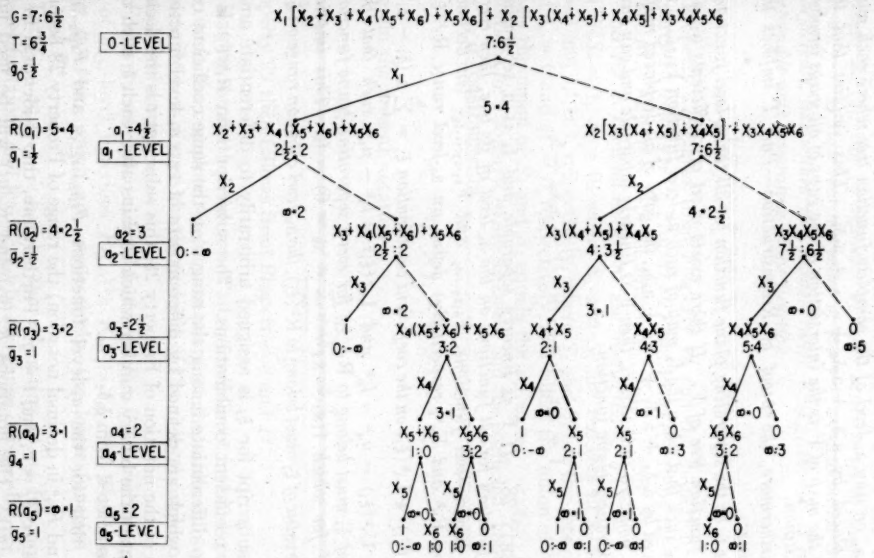
c. Necessary Properties—Part II

Reconstruction—Part I is a sufficient but not a necessary test.[12] If it can be completed the function has been proved linearly separable and a realization has been found. If it cannot, the issue is still in doubt, and further tests must be performed. This difficulty arises because Part I does not take into account all of the restrictions on the choice of the various coefficients. The range $\overline{R(a_k)}$ for example, gives the restrictions on a_k in order that all the a_{k-1}-level functions can have the same separating plane function, that is, $\overline{R(a_k)}$ is the range for proceeding one level up the tree. The constraints on a_k for proceeding to the very top of the tree may be somewhat more restrictive than $\overline{R(a_k)}$ and in fact the actual choice for a_k in Part I may be outside this more restrictive range. Thus when Part I "fails," that is, when on some a_k-level $\overline{R(a_k)}$ does not exist, it may be possible that a different choice for some of the previously selected coefficients, $a_{k+1}, \cdots a_n$, might yield a non-void range on a_k.

In this section these additional restrictions are studied resulting in a generalization of the concept of range.

For this we let the term *coefficient combination* represent any linear combination of coefficients. The following three properties define the generalized ranges and establish that they constitute the complete set of constraints on the coefficients. A sketch of the proof of these properties follows Property 29.

[12] It appears to be both necessary and sufficient for functions of less than about ten variables.


G = 7:6½
T = 6¾
q̄0 = ½
0-LEVEL
R(a1) = 5•4
q̄1 = ½
a1 = 4½
a1-LEVEL
R(a2) = 4•2½
q̄2 = ½
a2 = 3
a2-LEVEL
R(a3) = 3•2
q̄3 = 1
a3 = 2½
a3-LEVEL
R(a4) = 3•1
q̄4 = 1
a4 = 2
a4-LEVEL
R(a5) = ∞•1
q̄5 = 1
a5 = 2
a5-LEVEL
a6 = 1
X1 [X2 + X3 + X4(X5 + X6) + X5X6] + X2 [X3(X4 + X5) + X4X5] + X3X4X5X6
7:6½
X1
5•4
X2 + X3 + X4(X5 + X6) + X5X6
2½:2
X2
∞•2
X2 [X3(X4 + X5) + X4X5] + X3X4X5X6
7:6½
X2
4•2½
X3 + X4(X5 + X6) + X5X6
2½:2
X3
∞•2
X3(X4 + X5) + X4X5
4:3½
X3
3•1
X3X4X5X6
7½:6½
X3
∞•0
X4(X5 + X6) + X5X6
3:2
X4
3•1
X4 + X5
2:1
X4
∞•0
X4X5
4:3
X4
∞•1
X4X5X6
5:4
X4
∞•0
X5 + X6
1:0
X5X6
3:2
X5
∞•0
X5
∞•0
X5
∞•1
X5
∞•1
X5
∞•0
0:-∞
0:-∞
0:-∞
2:1
0:-∞
2:1
∞:3
0:-∞
3:2
∞:3
0:-∞
1:0
0:-∞
1:0
∞:1
0:-∞
∞:1
0:-∞
∞:1
1:0
∞:1
∞:5
0
0
0

PROPERTY 27. *Let* F *be linearly separable with separating plane function* f *and let* $f_k = f(x_1 = x_2 = \cdots = x_k = 0)$. *Then the map of* F *due to* f *is the union of the maps due to* f_k *of all the functions* $F_{\delta_1\delta_2\cdots\delta_k}$ *on any* a_k*-level of the complete function tree where each map is displaced by* $a_1\delta_1 + a_2\delta_2 + \cdots + a_k\delta_k$. *This implies that the gap of the map of* F *is the intersection of the gaps of all maps involved in the union.*

Furthermore, the proof of Reconstruction—Part I implies the following:

Let f_k *be the separating plane function for all the* a_k*-level functions of the function tree of* F. *If there exists a set of coefficients* $a_1, a_2, \cdots, a_k$ *such that when the* f_k *map of each* a_k*-level function* $F_{\delta_1\delta_2\cdots\delta_k}$ *is displaced by* $a_1\delta_1 + a_2\delta_2 + \cdots + a_k\delta_k$, *the gaps of all displaced maps have a non-zero intersection, then* F *is linearly separable and can have the separating plane function*

$$f = a_1x_1 + a_2x_2 + \cdots + a_kx_k + f_k$$

PROPERTY 28. *If* F *is linearly separable and* f_k *is a separating plane function for all functions on the* a_k*-level of its tree, then every set of two* a_k*-level functions,* $F_{\delta_1\delta_2\cdots\delta_k}$ *and* $F_{\delta_1'\delta_2'\cdots\delta_k'}$ *(in decimal notation* $F_p{}^k$ *and* $F_q{}^k$, *respectively) define an* a_k*-level range* $R_t{}^k(\xi_j) = U_t{}^k(\xi_j) \bigstar L_t{}^k(\xi_j)$ *on the coefficient combination* $\xi_j = \sum_{i=1}^{k} a_i(\delta_i - \delta_i')$ *where* $U_t{}^k(\xi_j) = u_q{}^k - l_p{}^k$ *and* $L_t{}^k(\xi_j) = l_q{}^k - u_p{}^k$ *such that the value of* ξ_j *must belong to* $R_t{}^k(\xi_j)$ *for every separating plane function* f *of* F *for which* $f(x_1 = x_2 = \cdots = x_k = 0) = f_k$. *This implies that the value of* ξ_j *must belong to* $\overline{R^k(\xi_j)}$, *the* a_k*-level common range on* ξ_j.

The subscript for ξ_j is assigned arbitrarily to differentiate among different coefficient combinations. The subscript t on $R_t{}^k(\xi_j)$ is included to differentiate among the ranges on the same coefficient combination which are defined by different sets of two a_k-level functions. In terms of the notation of Property 28, this subscript t is the decimal equivalent of the binary number which remains after each δ_i such that $\delta_i \neq \delta_i'$ is deleted from $\delta_1\delta_2\cdots\delta_k$.

Note that for the a_k-level functions $F_{\delta_1\delta_2\cdots\delta_{k-1}1}$ and $F_{\delta_1\delta_2\cdots\delta_{k-1}0}$ ($F_{2q+1}{}^k$ and $F_{2q}{}^k$ in decimal notation) the range of Property 28 becomes $R_q{}^k(a_k)$ since $\xi_j = a_k$ and $t = q$. Furthermore, this is identical to the range $R_q(a_k)$. The notations $R_q{}^k(a_k)$ and $R_q(a_k)$ will be used interchangeably. The common range $\overline{R^k(\xi_j)}$ is defined analogously to the common range $\overline{R(a_k)}$. The ranges for a_k on each a_k-level, that is, $R_q(a_k)$, or what is the same thing $R_q{}^k(a_k)$, we call the *direct ranges*. All others we term *indirect ranges*.

PROPERTY 29. *Let* f_k *be the separating plane function for all functions on any* a_k*-level of the function tree for* F. *Let* $\{\xi_j\}$ *denote the set of coefficient combinations on which* a_k*-level direct and indirect ranges* $R_i^k(\xi_j)$ *can be defined. If there exists a set of coefficients,* $a_1, a_2, \cdots, a_k$ *such that the value of every* ξ_j *as determined by these coefficients, lies within its common range* $\overline{R^k(\xi_j)}$, *then* F *is linearly separable with separating plane function*

$$f = a_1x_1 + a_2x_2 + \cdots + a_kx_k + f_k.$$

Property 27 follows from Property 14 and the induction theorem. Property 28 follows from Property 27 and Property 10 by the following argument. Any two a_k-level functions, $F_{\delta_1\delta_2\cdots\delta_k}$ and $F_{\delta_1'\delta_2'\cdots\delta_k'}$ (or $F_p{}^k$ and $F_q{}^k$ in decimal notation) are, respectively, $F(S_1 = 1, S_2 = 0)$ and $F(S_3 = 1, S_4 = 0)$ where S_1 and S_3 are the sets of x_i such that $\delta_i = 1$ and $\delta_i' = 1$, respectively, where S_2 and S_4 are the sets of x_i such that $\delta_i = 0$ and $\delta_i' = 0$, respectively, and where the union of S_1 and S_2 equals the union of S_3 and S_4 equals the set $x_1, x_2, \cdots, x_k$. Thus the conditions for Property 10 are satisfied, and the coefficient combination $s_1 - s_3$ is constrained by

$$u_q{}^k - l_p{}^k > s_1 - s_3 > l_q{}^k - u_p{}^k$$

wherein the present notation

$$s_1 - s_3 = \sum_{i=1}^{k} a_i(\delta_i - \delta_i').$$

Property 29 then follows from Properties 8 and 27.

Having shown that the set of all a_k-level direct and indirect ranges represents the complete set of constraints on the coefficients yet to be chosen, we now discuss the dependence of the a_k-level ranges on the set of a_{k+1}-level ranges.

For every set of two a_k-level functions $F_p{}^k$, $F_q{}^k$ there corresponds the set of a_{k+1}-level functions $F_{2p+1}{}^{k+1}$, $F_{2p}{}^{k+1}$, $F_{2q+1}{}^{k+1}$, $F_{2q}{}^{k+1}$, called the *lower two-pair*, which are, respectively, the lower function pairs of $F_p{}^k$ and $F_q{}^k$.

Let $f_k = a_{k+1}x_{k+1} + f_{k+1}$ be the separating plane function for all a_k-level functions and let $R_i{}^k(\xi_j)$ be the range defined by $F_p{}^k$ and $F_q{}^k$. Since in the structure of the tree, solid lines connect $F_p{}^k$ with $F_{2p+1}{}^{k+1}$ and $F_q{}^k$ with $F_{2q+1}{}^{k+1}$ and dotted lines connect $F_p{}^k$ with $F_{2p}{}^{k+1}$ and $F_q{}^k$ with $F_{2q}{}^{k+1}$, it follows from Property 27 that the functions of the lower two-pair provide direct ranges $R_p(a_{k+1})$ and $R_q(a_{k+1})$ defined by $F_{2p+1}{}^{k+1}$, $F_{2p}{}^{k+1}$ and $F_{2q+1}{}^{k+1}$, $F_{2q}{}^{k+1}$, respectively, as well as the following indirect

ranges. $F_{2p+1}{}^{k+1}$, $F_{2q+1}{}^{k+1}$ and $F_{2p}{}^{k+1}$, $F_{2q}{}^{k+1}$ define, respectively, $R_{2l+1}{}^{k+1}(\xi_j)$ and $R_{2l}{}^{k+1}(\xi_j)$. $F_{2p+1}{}^{k+1}$, $F_{2q}{}^{k+1}$ define $R_l{}^{k+1}(\xi_j + a_{k+1})$. $F_{2p}{}^{k+1}$, $F_{2q+1}{}^{k+1}$ define $R_l{}^{k+1}(\xi_j - a_{k+1})$.

These indirect ranges are called the a_{k+1}-level *implicant ranges* of $R_l{}^k(\xi_j)$. The upper and lower boundaries of these ranges are called, respectively, the a_{k+1}-level *upper and lower boundary implicants* of $R_l{}^k(\xi_j)$.

Consider any one of these implicant ranges, say $R_l{}^{k+1}(\xi_j + a_{k+1})$, together with the a_{k+1}-level functions which define it, in this case $F_{2p+1}{}^{k+1}$ and $F_{2q}{}^{k+1}$. Note that this range expresses a constraint on the coefficients which comprise $(\xi_j + a_{k+1})$ such that after the values for these coefficients are selected, the gaps of $F_{2p+1}{}^{k+1}$ and $F_{2q}{}^{k+1}$, displaced according to Property 27, will have a non-void intersection. Note moreover, that after the value for a_{k+1} is selected, the appropriately displaced gaps of $F_{2p+1}{}^{k+1}$ and $F_{2q}{}^{k+1}$ still impose a constraint on the coefficient combination ξ_j. This constraint has upper boundary $U_l{}^{k+1}(\xi_j + a_{k+1}) - A_{k+1}$ and lower boundary $L_l{}^{k+1}(\xi_j + a_{k+1}) - A_{k+1}$ where A_{k+1} represents the value selected for a_{k+1}. This constraint is denoted by $R_l{}^{k+1}(\xi_j + a_{k+1}) - A_{k+1}$. Similarly, the implicant ranges $R_l{}^{k+1}(\xi_j - a_{k+1})$, $R_{2l}{}^{k+1}(\xi_j)$ and $R_{2l+1}{}^{k+1}(\xi_j)$ impose constraints on ξ_j after the value of a_{k+1} is selected and these are given by $R_l{}^{k+1}(\xi_j - a_{k+1}) + A_{k+1}$, $R_{2l}{}^{k+1}(\xi_j)$ and $R_{2l+1}{}^{k+1}(\xi_j)$, respectively, where the latter two are independent of A_{k+1}. We call $R_l{}^{k+1}(\xi_j + a_{k+1}) - A_{k+1}$, $R_l{}^{k+1}(\xi_j - a_{k+1}) + A_{k+1}$, $R_{2l}{}^{k+1}(\xi_j)$ and $R_{2l+1}{}^{k+1}(\xi_j)$ the a_{k+1}-level *component ranges* of $R_l{}^k(\xi_j)$. The boundaries of the component ranges we call *component boundaries*.

We see, therefore, that each of the a_{k+1}-level component ranges of $R_l{}^k(\xi_j)$ expresses a necessary restriction on ξ_j after the value of a_{k+1} has been selected in order that the displaced gaps of $F_{2p+1}{}^{k+1}$, $F_{2p}{}^{k+1}$, $F_{2q+1}{}^{k+1}$ and $F_{2q}{}^{k+1}$ intersect by pairs. But intersection by pairs is a necessary and sufficient condition for complete intersection, hence, the component ranges express the restrictions on ξ_j after a_{k+1} has been selected in order that the displaced gaps of $F_{2p+1}{}^{k+1}$, $F_{2p}{}^{k+1}$, $F_{2q+1}{}^{k+1}$ and $F_{2q}{}^{k+1}$ have a non-void intersection. This is exactly the condition specified by $R_l{}^k(\xi_j)$, however; hence we have proved Property 30.

PROPERTY 30. *Let* $f_k = a_{k+1}x_{k+1} + f_{k+1}$ *be the separating plane function for all functions on the* a_k*-level of the function tree and let* $R_l{}^k(\xi_j)$ *be the range defined by any two* a_k*-level functions* $F_p{}^k$ *and* $F_q{}^k$. *If* A_{k+1} *is the value selected for* a_{k+1} *and if* A_{k+1} *belongs to* $\overline{R(a_{k+1})}$, *then* $R_l{}^k(\xi_j)$ *is the intersection of its* a_{k+1}*-level component ranges in the sense that*

$$R_l{}^k(\xi_j) = U_l{}^k(\xi_j) \bigstar L_l{}^k(\xi_j)$$

where

$$U_i^k(\xi_j) = Min \begin{Bmatrix} U_i^{k+1}(\xi_j - a_{k+1}) + A_{k+1} \\ U_i^{k+1}(\xi_j + a_{k+1}) - A_{k+1} \\ U_{2^k}^{k+1}(\xi_j) \\ U_{2^k+1}^{k+1}(\xi_j) \end{Bmatrix}, \qquad L_i^k(\xi_j) = Max \begin{Bmatrix} L_i^{k+1}(\xi_j - a_{k+1}) + A_{k+1} \\ L_i^{k+1}(\xi_j + a_{k+1}) - A_{k+1} \\ L_{2^k}^{k+1}(\xi_j) \\ L_{2^k+1}^{k+1}(\xi_j) \end{Bmatrix}.$$

Furthermore, the a_k*-level common range on* ξ_j, $\overline{R^k(\xi_j)}$, *is non-void if and only if each of its* a_{k+1}*-level common implicant ranges is non-void, that is,* $\overline{R^{k+1}(\xi_j)} \neq \phi$, $\overline{R^{k+1}(\xi_j + a_{k+1})} \neq \phi$ *and* $\overline{R^{k+1}(\xi_j - a_{k+1})} \neq \phi$, *and* a_{k+1} *is selected in* $\overline{R(a_{k+1})}$ *and such that the common* a_{k+1}*-level component ranges* $\overline{R^{k+1}(\xi_j)}$, $\overline{R^{k+1}(\xi_j + a_{k+1})} - A_{k+1}$ *and* $\overline{R^{k+1}(\xi_j - a_{k+1})} + A_{k+1}$ *have a non-void intersection.*

It follows from Property 30 that if for some coefficient combination ξ_j, $\overline{R^k(\xi_j)} = \phi$, then either one of its a_{k+1}-level common implicant ranges is void or the value selected for a_{k+1} is such that the a_{k+1}-level common component ranges do not intersect. These possibilities are investigated separately by Reconstruction—Parts II and III.

Part II determines whether an a_{k+1}-level common implicant range is void and, if so, whether one of its a_{k+2}-level common implicants is void and so on. If by this process a void a_{n-1}-level common implicant range is found, then the original switching function is not linearly separable according to Properties 24 and 30. If at any $a_{k+\nu}$-level all $a_{k+\nu}$-level implicants of $\overline{R^k(\xi_j)}$ are non-void, then Part III is used to determine whether a different choice for the coefficients $a_{k+\nu}$, $a_{k+\nu+1}$, $\cdots$, a_{n-1} will provide a non-void intersection of the appropriate common component ranges.

For each j, let the set of a_k-level ranges $R_i^k(\xi_j)$ be represented by $\{R^k(\xi_j)\}$ and let the sets of corresponding boundaries $U_i^k(\xi_j)$ and $L_i^k(\xi_j)$ be represented by $\{U^k(\xi_j)\}$ and $\{L^k(\xi_j)\}$, respectively. Each $U_i^k(\xi_j)$ such that $U_i^k(\xi_j) \leqslant \overline{L^k(\xi_j)}$ is called an *inconsistent upper boundary* and is relabeled $\mathfrak{U}_i^k(\xi_j)$. Similarly, each $L_i^k(\xi_j)$ such that $L_i^k(\xi_j) \geqslant \overline{U^k(\xi_j)}$ is called an *inconsistent lower boundary* and is relabeled $\mathfrak{L}_i^k(\xi_j)$. For each j, the sets of inconsistent boundaries are represented by $\{\mathfrak{U}^k(\xi_j)\}$ and $\{\mathfrak{L}^k(\xi_j)\}$, respectively, or by the set $[\{\mathfrak{U}^k(\xi_j)\}, \{\mathfrak{L}^k(\xi_j)\}]$. Each set is non-empty if and only if $\overline{R^k(\xi_j)} = \phi$. Those ranges $R_i^k(\xi_j)$ which have at least one inconsistent boundary are called inconsistent ranges and are relabeled $\mathfrak{R}_i^k(\xi_j)$.

d. Description of Reconstruction—Part II

Reconstruction—Part II begins on the a_k-level of the function tree where $\overline{R(a_k)} = \phi$ as determined by Reconstruction—Part I. Its purpose is to identify for $\nu = 1, 2, \cdots, n - 1 - k$ the inconsistent

$a_{k+\nu}$-level implicants of all $R_t(a_k)$. To do this we need only consider the boundary implicants of those boundaries of all $\mathfrak{R}_t(a_k)$. The description of Reconstruction—Part II is given in the following paragraphs. It is presented in a sequential format and is illustrated by Example 6.

(*i*) From the set $\{R(a_k)\}$ as determined by Part I, step (*iv*), select the set of inconsistent boundaries $[\{\mathfrak{U}(a_k)\}, \{\mathfrak{L}(a_k)\}]$.

Both boundary sets are non-void since $\overline{R(a_k)} = \phi$. Our objective, therefore, is to determine whether or not $\overline{R(a_k)} = \phi$ because some a_{k+1}-level implicant of these boundary sets is inconsistent. We proceed using the following general notation.

Let ν denote the number of levels between the a_k-level and the level on which ranges are being considered. Let ξ_j for $j = 1, 2, \cdots, \sigma_\nu$ denote the different coefficient combinations whose ranges on the $a_{k+\nu}$-level are being investigated.

To continue, let $\nu = 0$, $\sigma_\nu = \sigma_0 = 1$, and $\xi_j = \xi_1 = a_k$.

(*ii*) For each j where $j = 1, 2, \cdots, \nu$ determine from the inconsistent boundary set $[\{\mathfrak{U}^{k+\nu}(\xi_j)\}, \{\mathfrak{L}^{k+\nu}(\xi_j)\}]$ the three sets of $a_{k+\nu+1}$-level boundary implicants,[13]

$$[\{U^{k+\nu+1}(\xi_j)\}, \{L^{k+\nu+1}(\xi_j)\}],$$
$$[\{U^{k+\nu+1}(\xi_j + a_{k+\nu+1})\}, \{L^{k+\nu+1}(\xi_j + a_{k+\nu+1})\}]$$

and

$$[\{U^{k+\nu+1}(\xi_j - a_{k+\nu+1})\}, \{L^{k+\nu+1}(\xi_j - a_{k+\nu+1})\}].$$

In order to represent three sets with a simpler notation, let $\alpha = 1, 2, \cdots, 3\sigma_\nu$ and let $\zeta_\alpha = \xi_j$ for $\alpha = 3j$, $\zeta_\alpha = \xi_j + a_{k+\nu+1}$ for $\alpha = 3j - 1$, and $\zeta_\alpha = \xi_j - a_{k+\nu+1}$ for $\alpha = 3j - 2$. Using this notation the sets of $a_{k+\nu+1}$-level boundary implicants of step (*ii*) become $[\{U^{k+\nu+1}(\zeta_\alpha)\}, \{L^{k+\nu+1}(\zeta_\alpha)\}]$ for $\alpha = 1, 2, \cdots, 3\sigma_\nu$. We wish to determine whether any of these are inconsistent.

(*iii*) From each $[\{U^{k+\nu+1}(\zeta_\alpha)\}, \{L^{k+\nu+1}(\zeta_\alpha)\}]$ for $\alpha = 1, 2, \cdots, 3\sigma_\nu$ determine $\overline{U^{k+\nu+1}(\zeta_\alpha)}$ and $\overline{L^{k+\nu+1}(\zeta_\alpha)}$ according to

$$\overline{U^{k+\nu+1}(\zeta_\alpha)} = \text{smallest member of } \{U^{k+\nu+1}(\zeta_\alpha)\}$$

$$\overline{L^{k+\nu+1}(\zeta_\alpha)} = \text{largest member of } \{L^{k+\nu+1}(\zeta_\alpha)\}$$

[13] If any $\mathfrak{U}_i^{k+\nu}(\xi_j)$ or any $\mathfrak{L}_i^{k+\nu}(\xi_j)$ for $j = 1, 2, \cdots \sigma_{k+\nu}$ depends on a constant function then corresponding $a_{k+\nu+1}$-level boundary implicants will depend upon $a_{k+\nu+1}$-level function pairs which are constant and which are not included in the tree. When such pairs are required to complete step (*ii*) they must be supplied complete with gaps. Gaps for constant functions are described in Section II-A.

Although $[\{U^{k+\nu+1}(\zeta_\alpha)\}, \{L^{k+\nu+1}(\zeta_\alpha)\}]$ does not include all boundaries of all $a_{k+\nu+1}$-level ranges $R_l^{k+\nu+1}(\zeta_\alpha)$, it will include all boundaries which are inconsistent; therefore, it will include $\overline{U^{k+\nu+1}(\zeta_\alpha)}$ and $\overline{L^{k+\nu+1}(\zeta_\alpha)}$ when $\overline{U^{k+\nu+1}(\zeta_\alpha)} \leqslant \overline{L^{k+\nu+1}(\zeta_\alpha)}$.

(*iv*) Each $[\{U^{k+\nu+1}(\zeta_\alpha)\}, \{L^{k+\nu+1}(\zeta_\alpha)\}]$ for $\alpha = 1, 2, \cdots, 3\sigma_\nu$ is inconsistent if $\overline{U^{k+\nu+1}(\zeta_\alpha)} \leqslant \overline{L^{k+\nu+1}(\zeta_\alpha)}$. Arbitrarily relabel each $[\{U^{k+\nu+1}(\zeta_\alpha)\}, \{L^{k+\nu+1}(\zeta_\alpha)\}]$ which is inconsistent as $[\{U^{k+\nu+1}(\xi_j)\}, \{L^{k+\nu+1}(\xi_j)\}]$ where $j = 1, 2, \cdots, \sigma_{\nu+1}$ and where $\sigma_{\nu+1}$ denotes the number of inconsistent sets.

Note that the ξ_j of step (*iv*) and the ξ_j of step (*ii*) do not necessarily represent the same coefficient combination.

(*v*) (*a*) If $\nu + 1 = n - k - 1$ and if $\sigma_{\nu+1} > 0$, then the original switching function is not linearly separable.

(*b*) If $\nu + 1 < n - k - 1$ and if $\sigma_{\nu+1} > 0$, then go to step (*vi*).

(*c*) If $\nu + 1 \leq n - k - 1$ and if $\sigma_{\nu+1} > 0$, then denote the value of ν by ν_0 and go to Reconstruction—Part III.

Condition (*a*) implies some $\overline{R^{n-1}(\zeta_\alpha)} = \phi$. Since by Property 24 this condition is independent of the value of a_n, then according to Property 28 the original function is not linearly separable.

Condition (*b*) implies that one or more coefficient combinations ζ_α have void ranges on the $a_{k+\nu+1}$-level; hence Reconstruction—Part II is still applicable.

Condition (*c*) implies that all $a_{k+\nu+1}$-level implicants of all a_k-level ranges of $\{R(a_k)\}$ are non-void. Under this condition Part II is no longer applicable; however, the issue as to whether or not the original switching function is linearly separable is still unresolved. The answer is provided by Reconstruction—Part III. It makes use of the inconsistent boundary sets $[\{\mathfrak{U}^{k+\nu_0}(\xi_j)\}, \{\mathfrak{L}^{k+\nu_0}(\xi_j)\}]$ $j = 1, 2, \cdots, \sigma_{\nu_0}$ which were used in the previous step (*ii*) of Part II. These boundary sets are relabeled as $[\{\mathfrak{U}^{q+1}(\xi_j)\}, \{\mathfrak{L}^{q+1}(\xi_j)\}]$ $j = 1, 2, \cdots, v_0$ where $v_0 = \sigma_{\nu_0}$ when transferred to Part III.

(*vi*) From each $[\{U^{k+\nu+1}(\xi_j)\}, \{L^{k+\nu+1}(\xi_j)\}]$ for $j = 1, 2, \cdots, \sigma_{\nu+1}$ select the set of inconsistent boundaries

$$[\{\mathfrak{U}^{k+\nu+1}(\xi_j)\}, \{\mathfrak{L}^{k+\nu+1}(\xi_j)\}].$$

Increase the value of ν by unity and repeat the procedure starting with step (*ii*).

Note that the value of $k + \nu + 1$ in step (*vi*) is the same as the value of $k + \nu$ in the succeeding step (*ii*), since ν has been increased by unity.

EXAMPLE 6. Figure 7 shows the function tree for a six-variable function. Reconstruction—Part I has been used up to the a_3-level, where it could not be continued, that is, $R_1(a_2) = 2 \star 0$ and $R_0(a_2) = \infty \star 2$ and they do not intersect. Reconstruction—Part II will now be carried out. First the notation of the procedure will be applied to the present example.

1. Since the first non-existent range is on the a_3-level, $k = 2$.
2. Since there is only one non-existent a_2-level range, $\alpha_0 = 1$, and since that non-existent range is on a_2, $\xi_1^2 = a_2$.
3. The set $\{R(a_2)\}$ consists of all (both) a_2-level ranges on a_2, that is, the members of $\{R(a_2)\}$ are $2 \star 0$ and $\infty \star 2$.

The steps in the procedure are as follows:

i. The inconsistent boundary sets $\{\mathfrak{U}(a_2)\}$ and $\{\mathfrak{L}(a_2)\}$ each have only one member where

$$\{\mathfrak{U}(a_2)\} = 2 = u_2^2 - l_2^2 \qquad \text{and} \qquad \{\mathfrak{L}(a_2)\} = 2 = l_0^2 - u_1^2$$

and where the range boundaries are expressed in terms of a_2-level gap boundaries which have been numbered according to the function numbering.

ii. The sets of a_3-level boundary implicants are

(*a*) $[\{U^3(a_2)\}, \{L^3(a_2)\}]$ where

$\{U^3(a_2)\}$ has members		$\{L^3(a_2)\}$ has members
$\infty = u_8^3 - l_7^3$		$0 = l_8^3 - u_7^3$
$2 = u_4^3 - l_6^3$	and	$0 = l_4^3 - u_6^3$
$\infty = u_1^3 - l_3^3$		$2 = l_1^3 - u_3^3$
$\infty = u_0^3 - l_2^3$		$1 = l_0^3 - u_2^3$.

(*b*) $[\{U^3(a_2 + a_3)\}, \{L^3(a_2 + a_3)\}]$ where

$\{U^3(a_2 + a_3)\}$ has members		$\{L^3(a_2 + a_3)\}$ has members
$\infty = u_4^3 - l_7^3$	and	$1 = l_4^3 - u_7^3$
$\infty = u_0^3 - l_3^3$		$2 = l_0^3 - u_3^3$.

(*c*) $[\{U^3(a_2 - a_3)\}, \{L^3(a_2 - a_3)\}]$ where

$\{U^3(a_2 - a_3)\}$ has members		$\{L^3(a_2 - a_3)\}$ has members
$1 = u_5^3 - l_6^3$	and	$-1 = l_5^3 - u_6^3$
$\infty = u_1^3 - l_2^3$		$1 = l_1^3 - u_2^3$.

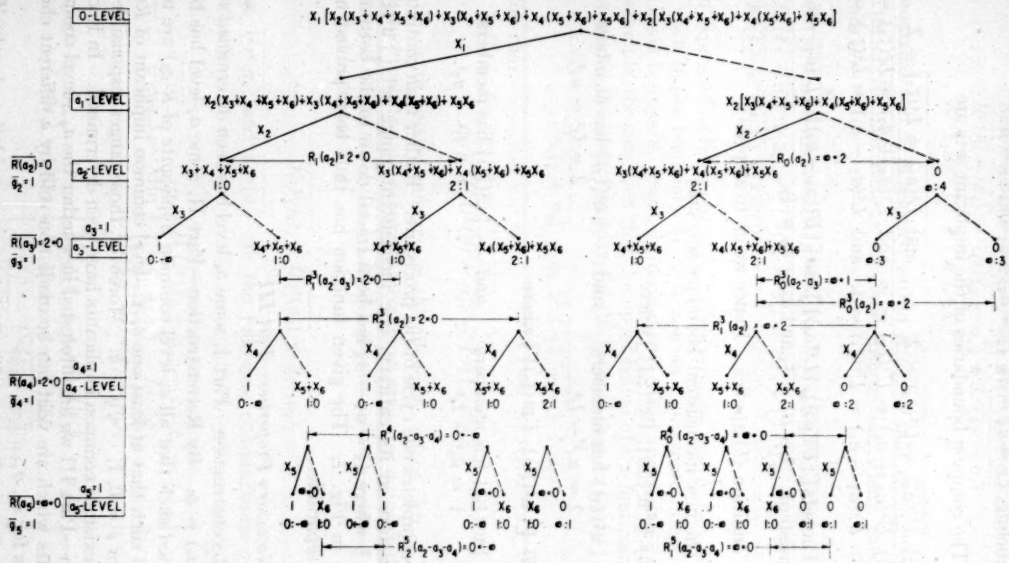


FIG. 7. Reconstruction—Part II for Example 6.

Re-labeling, $\zeta_1 = a_3 - a_2$, $\zeta_2 = a_3 + a_2$ and $\zeta_3 = a_2$.

iii. The common boundaries of the implicant sets are

$$(a)\ \overline{U^3(a_2)} = \overline{U^3(\zeta_3)} = 2 \quad \text{and} \quad \overline{L^3(a_2)} = \overline{L^3(\zeta_3)} = 2$$
$$(b)\ \overline{U^3(a_2 + a_3)} = \overline{U^3(\zeta_2)} = \infty \quad \text{and} \quad \overline{L^3(a_2 + a_3)} = \overline{L^3(\zeta_2)} = 2$$
$$(c)\ \overline{U^3(a_2 - a_3)} = \overline{U^3(\zeta_1)} = 1 \quad \text{and} \quad \overline{L^3(a_2 - a_3)} = \overline{L^3(\zeta_1)} = 1.$$

iv. The sets $[\{U^3(a_2)\}, \{L^3(a_2)\}]$ and $[\{U^3(a_2 - a_3)\}, \{L^3(a_2 - a_3)\}]$ are inconsistent. Let $a_2 = \xi_1$ and $a_2 - a_3 = \xi_2$.

v. Condition (*b*) applies. Hence we go to step (*vi*).

vi. The new inconsistent boundary sets are:

(*a*) $[\{\mathfrak{U}^3(\xi_1)\}, \{\mathfrak{L}^3(\xi_1)\}]$ where

$\{\mathfrak{U}^3(\xi_1)\}$ has members $2 = u_4^3 - l_6^3$ and $\{\mathfrak{L}^3(\xi_1)\}$ has members $2 = l_1^3 - u_3^3$.

(*b*) $[\{\mathfrak{U}^3(\xi_2)\}, \{\mathfrak{L}^3(\xi_2)\}]$ where

$\{\mathfrak{U}^3(\xi_2)\}$ has members $1 = u_5^3 - l_6^3$ and $\{\mathfrak{L}^3(\xi_2)\}$ has members $1 = l_1^3 - u_2^3$.

This completes one cycle of the procedure. Rather than continuing the procedure in its entirety, one inconsistent range set $[\{\mathfrak{U}^3(\xi_2)\}, \{\mathfrak{L}^3(\xi_2)\}]$, where $\xi_2 = a_2 - a_3$ has been traced down to the bottom of the tree in Fig. 7. The given function has thus been proved non-linearly separable.

e. Necessary Properties—Part III

By Reconstruction—Part I, some a_k-level has been determined such that $\overline{R(a_k)} = \phi$. By Reconstruction—Part II, some a_q-level has been determined such that all a_q-level common implicants of $\overline{R(a_k)}$ are non-void and such that at least one $a_{k+\nu+1}$-level common implicant of $\overline{R(a_k)}$ is void for $\nu = 0, 1, \cdots, q - k$. Moreover, those ranges responsible for the non-existent common implicants have been determined. In Reconstruction—Part III we are interested in whether the a_{q-1}-level common implicants which are void can be made non-void by a different choice of values for $a_q, a_{q+1}, \cdots, a_{n-1}$.

According to Property 30, $\overline{R^{q-1}(\xi_j)}$ is non-void if and only if its a_q-level implicants

$$\overline{R^q(\xi_j + a_q)} = \overline{U^q(\xi_j + a_q)} \star \overline{L^q(\xi_j + a_q)}$$
$$\overline{R^q(\xi_j - a_q)} = \overline{U^q(\xi_j - a_q)} \star \overline{L^q(\xi_j - a_q)}$$
$$\overline{R^q(\xi_j)} = \overline{U^q(\xi_j)} \star \overline{L^q(\xi_j)}$$

are non-void and a_q is selected within $\overline{R(a_q)}$ so that the a_q-level components of $\overline{R^{q-1}(\xi_j)}$ have a non-void intersection where these components are

$$\overline{R^q(\xi_j + a_q)} - a_q = [\overline{U^q(\xi_j + a_q)} - a_q] \star [\overline{L^q(\xi_j + a_q)} - a_q]$$
$$\overline{R^q(\xi_j - a_q)} + a_q = [\overline{U^q(\xi_j - a_q)} + a_q] \star [\overline{L^q(\xi_j - a_q)} + a_q]$$
$$\overline{R^q(\xi_j)} = \overline{U^q(\xi_j)} \star \overline{L^q(\xi_j)}$$

when expressed as functions of a_q. This requires that a_q be selected so that each upper boundary of each a_q-level component range exceeds each lower boundary. Such an inequality is called a *consistency relation*. For example, the condition that the upper bound of the first component range exceed the lower boundary of the second, that is, that

$$\overline{U^q(\xi_j + a_q)} - a_q > \overline{L^q(\xi_j - a_q)} + a_q$$

implies that

$$a_q < \tfrac{1}{2}[\overline{U^q(\xi_j + a_q)} - \overline{L^q(\xi_j - a_q)}]$$

and this is an upper boundary on the allowable values of a_q. The set of all such constraints defines the common range for a_q implied by the component ranges of $R_I^{q-1}(\xi_j)$. Such a range is called a *common implied range* and is denoted by

$$\overline{R_I^q(a_q)} = \overline{U_I^q(a_q)} \star \overline{L_I^q(a_q)}$$

where $\overline{U_I^q(a_q)}$ and $\overline{L_I^q(a_q)}$ are the following combinations of range boundaries

$$\overline{U_I^q(a_q)} = \text{Min} \left\{ \begin{array}{l} \tfrac{1}{2}[\overline{U^q(\xi_j + a_q)} - \overline{L^q(\xi_j - a_q)}] \\ \overline{U^q(\xi_j + a_q)} - \overline{L^q(\xi_j)} \\ \overline{U^q(\xi_j)} - \overline{L^q(\xi_j - a_q)} \end{array} \right\}$$

$$\overline{L_I^q(a_q)} = \text{Max} \left\{ \begin{array}{l} \tfrac{1}{2}[\overline{L^q(\xi_j + a_q)} - \overline{U^q(\xi_j - a_q)}] \\ \overline{L^q(\xi_j + a_q)} - \overline{U^q(\xi_j)} \\ \overline{L^q(\xi_j)} - \overline{U^q(\xi_j - a_q)}. \end{array} \right\}$$

These results are summarized by Property 31.

PROPERTY 31. *Each* $\overline{R^{q-1}(\xi_i)}$ *is non-void if and only if* (*i*) *all of its* a_q*-level components are non-void;* (*ii*) *the common range*[14] $\overline{R_I{}^q(a_q)}$ *implied by these components is non-void and intersects* $\overline{R(a_q)}$*; and* (*iii*) a_q *is selected within this intersection.*

If the a_q-level implied range on a_q does not exist or does not intersect the common range for a_q, it follows that no choice for a_q within the common range for a_q will remedy the inconsistency. We now investigate the possibility of remedying the inconsistency by a different choice for a_{q+1}, that is, we wish to determine the additional constraints which the value of a_{q+1} must satisfy in order that $\overline{R^{q-1}(\xi_j)}$ be consistent. Rather than determine the complete set of constraints on a_{q+1}, which would involve a tedious though straightforward computation, we determine a particular constraint called the *dominant implied range* which is defined below and which has the following properties: (1) It is a necessary restriction on the choice for a_{q+1}; (2) it is not satisfied by the current choice for a_{q+1}.

Consider the implied boundary set $[\{U_I{}^q(a_q)\}, \{L_I{}^q(a_q)\}]$ which is inconsistent.[15] Let $[\{\mathfrak{U}_I{}^q(a_q)\}, \{\mathfrak{L}_I{}^q(a_q)\}]$ denote the *inconsistent boundary set* composed of all upper boundaries which are less than or equal to $\overline{L_I{}^q(a_q)}$ and all lower boundaries which are greater than or equal to $\overline{U_I{}^q(a_q)}$. Each boundary in this boundary set can be expressed in terms of a_q-level gap boundaries.

Each a_q-level gap boundary has two a_{q+1}-level *gap boundary components*, such that the a_{q+1}-level gap boundary components of $u_p{}^q$ are $u_{2p+1}{}^{q+1} + A_{q+1}$ and $u_{2p}{}^{q+1}$. According to Properties 14 and 22

$$u_p{}^q = \text{Min} \begin{Bmatrix} u_{2p+1}{}^{q+1} + A_{q+1} \\ u_{2p}{}^{q+1} \end{Bmatrix}.$$

For the particular choice of a_{q+1} made previously, one (or possibly both) of these a_{q+1}-level components actually equals $u_p{}^q$. Those components having this property are called the a_{q+1}-level *dominant gap boundary components* of $u_p{}^q$. If in the set of dominant components, the value of A_{q+1} is replaced by a_{q+1} as a variable, the resultant set is called the a_{q+1}-level *dominant gap function boundary* of $u_p{}^q$. Note that these may or may not be functions of a_{q+1} depending upon whether the corresponding dominant gap boundaries contained A_{q+1}.

If each member of the inconsistent boundary set $[\{\mathfrak{U}_I{}^q(a_k)\},$

[14] This common range can be computed as above using the common implicant ranges, or the individual implicant ranges can be substituted into the consistency relation thus obtaining a set of implied ranges from which the common implied range can be determined in the usual manner.

[15] It will become apparent that this discussion applies to any a_q-level boundary set $[\{U^q(\xi_j)\}, \{L^q(\xi_i)\}]$ which is inconsistent.

$\{\mathfrak{L}_I{}^q(a_k)\}]$ is expressed in terms of its corresponding a_q-level gap boundaries and if these in turn are replaced by their a_{q+1}-level dominant gap function boundaries, then the set of consistency relations obtained by letting each member of $\{\mathfrak{U}_I{}^q(a_q)\}$ be greater than each member of $\{\mathfrak{L}_I{}^q(a_q)\}$ and denoted by

$$\{\mathfrak{U}_I{}^q(a_q)\} > \{\mathfrak{L}_I{}^q(a_q)\}$$

can be investigated as a function of a_{q+1}. The a_q-level consistency relations, when expressed in terms of the a_{q+1}-level dominant gap function boundaries, are a set of inequalities which must be satisfied. Those inequalities whose validity is a function of a_{q+1} determine upper and lower boundaries on a_{q+1} which together form the a_{q+1}-level *dominant implied range* on a_{q+1}.

Those inequalities whose validity is not a function of a_{q+1}, [for example, $(\ ) + a_{q+1} > [\ ] + a_{q+1}$, where $(\ )$ and $[\ ]$ denote sums and differences of a_{q+1}-level gap boundaries] determine other ranges called a_{q+1}-level *dominant residual ranges* which must be satisfied. In the above example the residual range is determined by $(\ ) > [\ ]$ and is a range on $a_q - a_{q+1}$.

Since the given a_q-level ranges were inconsistent it follows that either

1. One or more of the a_{q+1}-level dominant implied or residual ranges is inconsistent

or

2. The previous choice for a_{q+1} was not inside its dominant implied range.

This discussion is summarized in Property 32.

PROPERTY 32. (*a*) *In order for* $\overline{R^{q-1}(\xi_i)}$ *to be non-void, it is necessary that the* a_{q+1}*-level dominant implied range on* a_{q+1} *and the dominant residual ranges all be consistent, that the dominant implied range on* a_{q+1} *intersect the direct range on* a_{q+1} *and that* a_{q+1} *be selected inside this dominant implied range.*
(*b*) *If* $R^{q-1}(\xi_i)$ *is void, and its* a_q*-level implied range on* a_q *is void but all its* a_q*-level implicant ranges are consistent, then either*

(*1*) *One or more of the* a_{q+1}*-level dominant implied or residual ranges is inconsistent*

or

(*2*) *The previous choice for* a_{q+1} *was not inside the intersection of the dominant implied range and the direct range.*

For an example of the computation of dominant ranges see step (*c*) of Examples 7 and 8.

f. Description of Reconstruction—Part III

Part III begins on the a_{q-1}-level where it is known from Part II that some group of boundary sets $[\{\mathfrak{U}^{q-1}(\xi_j)\}, \{\mathfrak{L}^{q-1}(\xi_j)\}]$ $j = 1, 2, \cdots, v_0$ are inconsistent, but that all of their a_q-level implicant range sets are consistent. The object of Part III is to see whether any different choices for $a_{q+\eta}$, $\eta = 0, 1, \cdots, n-q-1$ would allow all the given boundary sets to be consistent. The procedure is described in sequential format and illustrated by Examples 7 and 8. To begin, let $\eta = 0$.

(*i*) For each j, where $j = 1, 2, \cdots, v_\eta$, there is an $a_{q+\eta-1}$-level inconsistent boundary set denoted by $[\{\mathfrak{U}^{q+\eta-1}(\xi_j)\}, \{\mathfrak{L}^{q+\eta-1}(\xi_j)\}]$.

(*a*) Express each upper and lower range boundaries of each $[\{\mathfrak{U}^{q+\eta-1}(\xi_j)\}, \{\mathfrak{L}^{q+\eta-1}(\xi_j)\}]$ in terms of corresponding $a_{q+\eta-1}$-level gap boundaries.

(*b*) Express each $a_{q+\eta-1}$-level boundary of (*a*) in terms of its $a_{q+\eta}$-level dominant gap function boundary or boundaries.

(*c*) For each $[\{\mathfrak{U}^{q+\eta-1}(\xi_j)\}, \{\mathfrak{L}^{q+\eta-1}(\xi_j)\}]$ express the consistency relations

$$\{\mathfrak{U}^{q+\eta-1}(\xi_j)\} > \{\mathfrak{L}^{q+\eta-1}(\xi_j)\}$$

in terms of the $a_{q+\eta}$-level dominant gap function boundaries of (*b*).

(*d*) From the resultant inequalities of (*c*) whose validity depends on the value of $a_{q+\eta}$, solve for upper and/or lower boundaries on $a_{q+\eta}$. These boundaries determine the boundary set of the dominant implied range on $a_{q+\eta}$ denoted by $[\{U_{I(\xi_j)^{q+\eta}}(a_{q+\eta})\}, \{L_{I(\xi_j)^{q+\eta}}(a_{q+\eta})\}]$.

(*e*) The $a_{q+\eta}$-level dominant residual ranges come from those inequalities of (*c*) whose validity is not dependent on the value of $a_{q+\eta}$. For each j these inequalities are of the form

$$(\ )_{\alpha j} + K_{\alpha j} a_{q+\eta} > [\ ]_{\alpha j} + K_{\alpha j} a_{q+\eta}$$

where $(\ )_{\alpha j}$ and $[\ ]_{\alpha j}$ are sums and differences of $a_{q+\eta}$-level gap boundaries and where $K_{\alpha j}$ can have only zero or integer values. All inequalities with the same j and the same value of $K_{\alpha j}$ form one $a_{k+\eta}$-level dominant residual boundary set where the corresponding $\{(\ )_{\alpha j}\}$ and $\{[\ ]_{\alpha j}\}$ are the upper and lower boundaries. We denote these $a_{q+\eta}$-level boundary sets by

$$[\{U_R{}^{q+\eta}(\zeta_i)\}, \{L_R{}^{q+\eta}(\zeta_i)\}].$$

(*ii*) From (*i*)–(*d*), determine the boundary set $[\{U_I^{q+\eta}(a_{q+\eta})\}, \{L_I^{q+\eta}(a_{q+\eta})\}]$, where[16]

$$\{U_I^{q+\eta}(a_{q+\eta})\} = \bigcup_{\xi_j} \{U_{I(\xi j)}^{q+\eta}(a_{q+\eta})\}$$

$$\{L_I^{q+\eta}(a_{q+\eta})\} = \bigcup_{\xi_j} \{L_{I(\xi j)}^{q+\eta}(a_{q+\eta})\}$$

and where the unions are taken over all $j = 1, 2, \cdots, \nu_\eta$.

$\{U_I^{q+\eta}(a_{q+\eta})\}$ and $\{L_I^{q+\eta}(a_{q+\eta})\}$ are the boundary sets for the range on $a_{q+\eta}$ implied by all the $a_{q+\eta-1}$-level inconsistent boundary sets of step (*i*)–(*c*). We now wish to determine whether this implied range or any of the residual ranges are inconsistent.

(*iii*) For each residual boundary set $[\{U_R^{q+\eta}(\zeta_i)\}, \{L_R^{q+\eta}(\zeta_i)\}]$ as determined by step (*i*)–(*e*) determine the common upper and lower boundaries $\overline{U_R^{q+\eta}(\zeta_i)}$ and $\overline{L_R^{q+\eta}(\zeta_i)}$ according to

$$\overline{U_R^{q+\eta}(\zeta_i)} = \text{smallest member of } \{U_R^{q+\eta}(\zeta_i)\}$$

$$\overline{L_R^{q+\eta}(\zeta_i)} = \text{largest member of } \{L_R^{q+\eta}(\zeta_i)\}.$$

Similarly determine the common boundaries $\overline{U_I^{q+\eta}(a_{q+\eta})}$ and $\overline{L_I^{q+\eta}(a_{q+\eta})}$ for the implied boundary set $[\{U_I^{q+\eta}(a_{q+\eta})\}, \{L_I^{q+\eta}(a_{q+\eta})\}]$ as determined by step (*ii*).

(*iv*) Determine the boundary sets of step (*iii*) which are inconsistent. A boundary set $[\{U^{q+\eta}(\ )\}, \{L^{q+\eta}(\ )\}]$ is inconsistent if $\overline{U^{q+\eta}(\ )} < \overline{L^{q+\eta}(\ )}$.

(*v*) (*a*) If $\eta = n - 1 - q$ and if from step (*iv*) either the $a_{q+\eta}$-level implied boundary set or any $a_{q+\eta}$-level residual boundary set is inconsistent the original function is not linearly separable.

(*b*) If $\eta < n - 1 - q$ let ρ denote the number of $a_{q+\eta}$-level residual boundary sets of step (*iv*) which are inconsistent. If $\rho > 0$ go to step (*vi*). Otherwise go to (*c*).

(*c*) If $\eta < n - 1 - q$ and if the $a_{q+\eta}$-level implied boundary set of step (*iv*) is inconsistent, go to step (*viii*). Otherwise go to (*d*).

(*d*) If the $a_{q+\eta}$-level implied range for $a_{q+\eta}$ of step (*iii*) is consistent, go to step (*ix*).

[16] $\bigcup$ denotes union.

Condition (*a*) implies that some a_{n-1}-level boundary set is inconsistent. Since by Property 24 this condition is independent of the value of a_n, then by Property 28, the original function is not linearly separable.

Conditions (*b*) and (*c*) both imply that there is an inconsistency which cannot be remedied by the choice for $a_{q+\eta}$, but that, since $q + \eta < n - 1$, there are still some coefficients left to be checked; therefore, Part III is still applicable.

Condition (*d*) implies that the implied range for $a_{q+\eta}$ exists and suggests that perhaps a different choice for $a_{q+\eta}$ would remedy the inconsistency.

(*vi*) Arbitrarily relabel each inconsistent residual boundary set $[\{U_R{}^{q+\eta}(\zeta_i)\}, \{L_R{}^{q+\eta}(\zeta_i)\}]$ of step (*iv*) as $[\{U_R{}^{q+\eta}(\xi_j)\}, \{L_R{}^{q+\eta}(\xi_j)\}]$ where $j = 1, 2, \cdots, \rho$.

(*vii*)[17] For each $[\{U_R{}^{q+\eta}(\xi_j)\}, \{L_R{}^{q+\eta}(\xi_j)\}]$ of step (*vi*) determine the boundaries which are inconsistent and denote these by $[\{\mathfrak{U}^{q+\eta}(\xi_j)\}, \{\mathfrak{L}^{q+\eta}(\xi_j)\}]$ where $j = 1, 2, \cdots, \rho$. Return to step (*v*).

(*viii*)[18] From the inconsistent implied boundary set $[\{U_I{}^{q+\eta}(a_{q+\eta})\}, \{L_I{}^{q+\eta}(a_{q+\eta})\}]$ of step (*iv*) determine the boundaries which are inconsistent and denote these by $[\{\mathfrak{U}^{q+\eta}(\xi_j)\}, \{\mathfrak{L}^{q+\eta}(\xi_j)\}]$ where $j = \rho + 1$. Let $v_{q+1} = \rho + 1$, increase the value of η by unity[19] and return to step (*i*).

(*ix*) Determine whether the common direct range for $a_{q+\eta}$ from Part I which is $\overline{R(a_{q+\eta})} = \overline{U(a_{q+\eta})} \star \overline{L(a_{q+\eta})}$, intersects the $a_{q+\eta}$ as determined in step (*iii*), that is, determine whether or not

$$\overline{U(a_{q+\eta})} > \overline{L_I{}^{q+\eta}(a_{q+\eta})}$$

and

$$\overline{U_I{}^{q+\eta}(a_{q+\eta})} > \overline{L(a_{q+\eta})}.$$

(*x*)[17-20] (*a*) If in step (*ix*) there is an intersection, go to (*b*); otherwise append all $U_i(a_{q+\eta})$ such that $U_i(a_{q+\eta}) = \overline{U(a_{q+\eta})}$ and all

[17] Step (*v*)-(*b*) transfers to this step.

[18] Step (*v*)-(*c*) transfers to this step.

[19] Note that the value of $q + \eta$ in step (*viii*) is the same as the value of $q + \eta - 1$ in the succeeding step (*i*), since η has been increased in value by unity.

[20] Step (*v*)-(*d*) transfers to this step.

FIG. 8. Reconstruction—Part III for Exampl 7.

$L_t(a_{q+\eta})$ such that $L_t(a_{q+\eta}) = \overline{L(a_{q+\eta})}$ to the set $[\{U_t^{q+\eta}(a_{q+\eta})\}, \{L_t^{q+\eta}(a_{q+\eta})\}]$ and represent the resultant boundary set by $[\{\mathfrak{U}^{q+\eta}(\xi_j)\}, \{\mathfrak{L}^{q+\eta}(\xi_j)\}]$ where $j = \rho + 1$.

(α) If $\eta = n - 1 - q$, then the original function is not linearly separable.

(β) If $\eta < n - 1 - q$, then let $v_{\eta+1} = \rho + 1$, increase the value of η by unity[21] and return to step (i).

(b) If the intersection of step (ix) exists and if $\rho = 0$, then this intersection is the new $a_{q+\eta}$-level common range for $a_{q+\eta}$ which we denote by $\overline{R(a_{q+\eta})}$ and return to step (iv)—Part I. If the intersection of step (ix) exists and if $\rho > 0$ then let $v_{\eta+1} = \rho$, increase the value of η by unity[21] and return to step (i).

Condition (a) implies that an $a_{q+\eta}$-level inconsistent boundary set exists and that Part III is still applicable.

Condition (b) implies that we have found a new range for $a_{q+\eta}$ which will remedy the inconsistencies under consideration. Hence we call this the new $a_{q+\eta}$-level common range for $a_{q+\eta}$, $\overline{R(a_{q+\eta})}$, and return to Part I.

This completes the discussion of one cycle of the procedure. The cycle begins on the a_{q-1}-level where an inconsistent boundary set exists. At the end of the cycle either a new value for a_q has been found which will remedy the inconsistency (that is, which will get us back to the a_{q-1}-level) or it has been proved that no value of a_q in the range of a_q will remedy the inconsistency and we repeat the procedure on the a_q-level. In the latter case if we are at the bottom of the tree, that is, if $q = n - 1$, the function has been proved non-linearly separable.

EXAMPLE 7. Figure 8 shows for a 13-variable function, the a_3- and a_4-level gaps and direct ranges which result from Reconstruction—Part I. At this point $a_4 = 11\frac{1}{2}$ and $\overline{R(a_3)} = \phi$. The inconsistent range-boundary set is $[\{\mathfrak{U}^3(a_3)\}, \{\mathfrak{L}^3(a_3)\}]$ where

$\{\mathfrak{U}^3(a_3)\}$ has members
$U_3^3(a_3) = 12\frac{1}{2}$

and

$\{\mathfrak{L}^3(a_3)\}$ has members
$L_0^3(a_3) = 12\frac{1}{2}$
$L_2^3(a_3) = 12\frac{1}{2}$.

It is readily verified using Part II that the corresponding a_4-level implicant boundary sets are all consistent; hence Part III is necessary.

[21] Note that the value of $q + \eta$ in step (x) is the same as the value of $q + \eta - 1$ in the succeeding step (i), since η has been increased in value by unity.

(*i*) (*a*) The a_3-level gap boundary expressions for $\{\mathfrak{U}^3(a_3)\}$ are

$$U_3{}^3(a_3) = u_6{}^3 - l_7{}^3$$

and for $\{\mathfrak{L}^3(a_3)\}$ are

$$L_0{}^3(a_3) = l_0{}^3 - u_1{}^3$$
$$L_2{}^3(a_3) = l_4{}^3 - u_5{}^3.$$

(*b*) The corresponding a_4-level dominant gap function boundaries are the following:

$$u_6{}^3 = 24 = u_{12}{}^4 = 24,$$
$$l_7{}^3 = 11\tfrac{1}{2} = l_{15}{}^4 + a_4 = 0 + a_4,$$

thus

$$u_0{}^3 - l_7{}^3 = u_{12}{}^4 - l_{15}{}^4 - a_4 = 24 + a_4.$$
$$l_0{}^3 = 52\tfrac{1}{2} = l_1{}^4 + a_4 = 41 + a_4$$
$$u_1{}^3 = 40 = u_2{}^4 = 40,$$

thus

$$l_0{}^3 - u_1{}^3 = l_1{}^4 - u_2{}^4 + a_4 = 1 + a_4.$$

Similarly,

$$l_4{}^3 - u_5{}^3 = l_9{}^4 - u_{10}{}^4 + a_4 = 1 + a_4.$$

(*c*) The two simultaneous inequalities are identical and are

$$24 - a_4 > 1 + a_4.$$

(*d*) The a_4-level dominant implied range is

$$R_{I(a_3)^4}(a_4) = 11\tfrac{1}{2} \bigstar - \infty.$$

(*e*) There are no residual ranges.

(*ii*) The boundary set $[\{U_I{}^4(a_4)\}, \{L_I{}^4(a_4)\}]$ is as follows:

$\{U_I{}^4(a_4)\}$ has members

1. $11\tfrac{1}{2} = \tfrac{1}{2}[24 - 1]$
$= \tfrac{1}{2}[u_{12}{}^4 - l_{15}{}^4 - l_1{}^4 - u_2{}^4]$

2. $11\tfrac{1}{2} = \tfrac{1}{2}[24 - 1]$
$= \tfrac{1}{2}[u_{12}{}^4 - l_{15}{}^4 - l_9{}^4 + u_{10}{}^4]$

and $\{L_I{}^4(a_4)\}$ has no members.

(*iii*) *and* (*iv*) $R_I^4(a_4)$ is obviously consistent.

(*v*) Condition (*d*) applies, since the implied and residual ranges are all consistent. Hence we continue at step (*ix*).

(*ix*) Since the common range $\overline{R(a_4)} = 12 \star 11$, the intersection of $\overline{R_I^4(a_4)}$ and $\overline{R(a_4)}$ is $11\frac{1}{2} \star 11$.

(*x*) Condition (*b*) applies. Let $\overline{R(a_4)} = 11\frac{1}{2} \star 11$, and return to step (*iv*), Part I. That step will select $a_4 = 11\frac{1}{4}$.

It can now be easily verified that after selecting $a_4 = 11\frac{1}{4}$, Reconstruction—Part I can be continued to the top of the tree selecting $a_3 = 12\frac{1}{2}$, $a_2 = 13\frac{3}{8}$ and $a_1 = 15\frac{1}{8}$, thus completing the realization of the function.

EXAMPLE 8. Figure 9 shows the gaps and direct ranges on the a_2- and a_3-levels of the tree of a 12-variable function after Reconstruction—Part I has been applied. This particular function, due to E. F. Moore of Bell Telephone Laboratories, was originally proposed as a counter-example to a conjecture of Stabler, Paull and McCluskey, and was mentioned in a letter written by the latter two and referenced in Section II-*C*-*a*. Space does not allow the complete test to be included here but it is included in (2).

The situation depicted here occurs near the end of the test. Proceeding up the tree using Reconstruction—Part I, the common range for a_3, $\overline{R(a_3)}$, is found to be $2 \star 1\frac{3}{4}$ and a_3 is selected equal to $1\frac{7}{8}$. On the next level $\overline{R(a_2)}$ is found to be void.

The inconsistent boundary set is $[\{\mathfrak{U}^2(a_2)\}, \{\mathfrak{L}^2(a_2)\}]$ where

$\{\mathfrak{U}^2(a_2)\}$ has member $U_0^2(a_2) = 1\frac{3}{8}$ and $\{\mathfrak{L}^2(a_2)\}$ has member $L_1^2(a_2) = 1\frac{3}{4}$.

It is readily verified using Reconstruction—Part II, that all of the a_2-level implicant boundary sets of $U_0^2(a_2)$ and $L_1^2(a_2)$ are consistent; hence Part III is necessary.

(*i*) (*a*) The a_2-level gap boundary expressions for $\{\mathfrak{U}^2(a_2)\}$ are $U_0^2(a_2) = 1\frac{3}{8} = u_0^2 - l_1^2$, and for $\{\mathfrak{L}^2(a_2)\}$ are $L_1^2(a_2) = 1\frac{3}{4} = l_2^2 - u_3^2$.

(*b*) The a_3-level dominant gap function boundaries are $u_0^2 - l_1^2 = u_0^3 - l_3^3 - a_3 = 3\frac{1}{4} - a_3$ and $l_2^2 - u_3^2 = l_5^3 + a_3 - u_7^3 - a_3 = 1\frac{3}{4}$.

(*c*) The simultaneous inequalities are $u_0^3 - l_3^3 - a_3 > l_5^3 - u_7^3$ or $3\frac{1}{4} - a_3 > 1\frac{3}{4}$.

FIG. 9. Reconstruction—Part III for Example 8.

(*d*) The resultant a_3-level dominant implied range boundary is $a_3 < u_0^3 + u_7^3 - l_3^3 - l_5^3$ or $a_3 < 1\frac{1}{2}$, and the a_3-level dominant implied range is $R_I^3(a_3) = 1\frac{1}{2} \star -\infty$.

(*e*) There is no dominant residual range.

(*ii*) The boundary set $[\{U_I^3(a_3)\}, \{L_I^3(a_3)\}]$ is as follows:

$\{U_I^3(a_3)\}$ has members
$u_0^3 - l_3^3 - l_5^3 + u_7^3 = 1\frac{1}{2}$,
and $\{L_I^3(a_3)\}$ has no members.

(*iii*) *and* (*iv*) The a_3-level implied boundary set is consistent.

(*v*) Condition (*d*) is satisfied; hence we continue at step (*ix*).

(*ix*) The common range $\overline{R(a_3)}$ does not intersect the implied range $R_I^3(a_3)$ since

$$\overline{R(a_3)} = 2 \star 1\tfrac{3}{4}$$
$$R_I^3(a_3) = 1\tfrac{1}{2} \star \infty.$$

(*x*) Condition (*a*) is satisfied. The new implied boundary set is as follows:

$\{U_I^3(a_3)\}$ has members
$u_0^3 - l_3^3 - l_5^3 + u_7^3 = 1\frac{1}{2}$
$u_0^3 - l_1^3 = 2$
$u_2^3 - l_3^3 = 2$
$u_4^3 - l_5^3 = 2.$

and

$\{L_I^3(a_3)\}$ has members
$l_6^3 - u_7^3 = 1\frac{3}{4}$

Relabeling, a_3 becomes ξ_1. The a_3-level inconsistent boundary set $[\{\mathfrak{U}^3(a_3)\}, \{\mathfrak{L}^3(a_3)\}]$ is as follows:

$\mathfrak{U}^3(a_3)$ has members
$1\frac{1}{2} = u_0^3 - l_3^3 - l_5^3 + u_7^3$

and

$\mathfrak{L}^3(a_3)$ has members
$1\frac{3}{4} = l_6^3 - u_7^3$

Condition (β) is satisfied. We continue at step (*i*).

This completes one cycle of the procedure. Starting with an inconsistent boundary set on the a_3-level, it has been shown that no value of a_3 in the range of a_3 will remedy the inconsistency, and an inconsistent a_3-level boundary set has been found with which to start the next cycle. If the procedure is continued, inconsistent dominant residual ranges appear on the next level and in general the computations become more tedious but eventually the bottom of the tree is reached thus proving the given function non-linearly separable.

g. Discussion of Reconstruction

If the procedure terminates at the top of the tree the function has been proved linearly separable and a realization has been obtained. If it terminates at the bottom of the tree the function has been proved non-linearly separable since a fundamental inconsistency has been found among the truth-table points (on the bottom level of the tree). We will now show that the procedure will always terminate at either the top or the bottom of the tree (that is, it will not oscillate indefinitely up and down the tree). From this it follows that the procedure is in fact a necessary and sufficient test.

Assume that the procedure does not terminate but that it oscillates indefinitely, never reaching the top or the bottom of the tree. There must then be a lowest level of the tree to which it returns repeatedly in the course of its oscillations. Call this the a_l-level. Each time it returns to this level, there is some inconsistency among a_l-level ranges which can be remedied by a different choice for a_l. The previous range for a_l was the intersection of the ranges produced by all the range constraints previously checked. Thus the previous choice for a_l satisfied all of the range constraints previously checked. The fact that this choice for a_l is not adequate but that there is another choice for a_l which is adequate implies that there is an a_l-level range constraint that has not yet been checked. But there are only a finite number of pertinent a_l-level range constraints. Thus the procedure can not return repeatedly to the a_l-level and the desired property is proved.

PROPERTY 33. *The test consisting of Decomposition and Reconstruction Parts I, II and III is both necessary and sufficient.*

The description of the test makes it appear more difficult than it actually is. The notation is complicated by the necessity of accommodating certain situations that occur only rarely. If the test is carried out a few times in simple situations so that its logic becomes clear, much of the notation can be dispensed with. In addition certain modifications can be made in the procedure, which are obvious after one understands the test, but whose description would require even more complications in the notation. This is discussed in (2).

The time for performing the test is of course dependent upon whether Reconstruction—Parts II and III are needed, and if they are, on the number of iterations. Functions of about eight variables (for which II and III are never needed) can be done in about a half hour of hand computations. Moore's twelfth-order example (Example 8) which is somewhat simplified by reason of its being highly symmetric in many of its variables, can be done in a little over four hours of hand computation. Space does not allow the inclusion of any complete tests of functions of this many variables, but such a test is included in (2).

The following questions arise from this procedure and remain unanswered:

1. For what number of variables does the existence of a function tree in which function pairs have the covering property imply that the switching function is linearly separable?

2. For what number of variables does the failure of Reconstruction—Part I imply that the switching function is not linearly separable?

3. Can the procedure be shortened when it is known that the switching function is symmetric in certain variables and that, if it is linearly separable, it has a separating plane function wherein all of the corresponding coefficients are equal?

Acknowledgments

The authors would like to express their appreciation to F. C. Hennie of the Massachusetts Institute of Technology and J. Hartmanis and R. L. Shuey of this Laboratory for valuable criticism and suggestions.

REFERENCES

(1) C. K. CHOW, "Boolean Functions Realizable with Single Threshold Devices," *Proc. I.R.E.*, Vol. 49, pp. 370–1 (1961).
(2) C. L. COATES AND P. M. LEWIS, II, "Linearly Separable Switching Functions," General Electric Research Lab. Report No. 61–RL–2764E, Schenectady, N. Y.
(3) C. G. ELGOT, "Truth Functions Realizable by Single Threshold Organ," *Proc. N.E.C.*, Vol. 16, p. 331–2 (1960).
(4) R. MCNAUGHTON, "Unate Truth Functions," Stanford University, Applied Mathematics and Statistics Laboratory, Technical Report No. 4, October 21, 1957; also *Proc. I.R.E., P.G.E.C.*, Vol. EC-10, pp. 1–6 (1961).
(5) R. C. MINNICK, "Synthesis of Linear Input Logic by the Simplex Method," Sixth Annual Symposium on Computers and Data Processing of the Denver Research Institute. July, 1959; "Linear Input Logic," *Proc. I.R.E., P.G.E.C.*, Vol. EC-10, pp. 6–16 (1961),
(6) E. F. MOORE, "Counterexample to a Conjecture of McCluskey and Paull," 1957 (unpublished).
(7) S. MUROGA, "The Principle of Majority Decision Logical Elements and the Complexity of Their Circuits," *Proc. Internatl. Conf. Information Processing*, Paris, UNESCO House, 1959, pp. 400–407.
(8) M. C. PAULL AND E. J. MCCLUSKEY, JR., "Boolean Functions Realizable with Single Threshold Devices," *Proc. I.R.E.*, Vol. 48, pp. 1335–1337 (1960).
(9) O. STRAM, "Arbitrary Boolean Functions of N Variables Realizable in Terms of Threshold Devices," *Proc. I.R.E.*, Vol. 49, p. 210–220 (1961).
(10) R. O. WINDER, "Single Stage Threshold Logic," *Proc. N.E.C.*, Vol. 16, p. 333 (1960).
(11) R. O. WINDER, "Some Recent Papers in Threshold Logic," *Proc. I.R.E.*, p. 1100(1961).

HIGH-TEMPERATURE HEAT CAPACITY OF DIAMOND

Recent measurements by Andrew Victor of the National Bureau of Standards have resulted in improved values for the high-temperature heat capacity of diamond. These data, obtained over the range of 323 to 1073 °K, and accurate to within 0.5 per cent over most of the range, join smoothly with other values down to the temperature of liquid hydrogen. An experimental basis is thus provided for the verification of theoretical treatments of diamond lattice dynamics over a wide range of temperatures.

Diamond-type crystals are of considerable interest for several reasons. Because the "low temperature" behavior of the diamond extends to nearly 1000 °K, useful data can be obtained with both low- and high-temperature calorimeters. Also, as the interatomic forces in the diamond lattice are largely covalent, continued investigation should provide further information concerning the directional characteristics of these bonds. Furthermore, silicon and germanium, materials of increasing scientific and industrial importance, possess crystal structures similar to the diamond.

The usual procedure for solving problems concerning diamond lattice dynamics is to propose a model of the interatomic forces and, employing the elastic constants and first-order Raman frequency shift, to solve a series of equations leading to a calculated value of heat capacity. In order to check the results of such a treatment, reliable values of measured heat capacity must be available.

The last reported measurements of the heat capacity of diamond at high temperatures were made in 1926. The values obtained at that time are believed to be in error because of oxidation of the sample during the measurements. Precautions were taken to prevent oxidation of the diamonds in the present work, and the data obtained by the Bureau are believed to be more reliable than previous results in the temperature range employed.

It is well known that diamonds can be formed by subjecting graphite to extremely high temperatures and pressures. A combination of several types of measurements, including heat capacity, heat of formation compressibility, and thermal expansion studies on graphite and diamond permits scientists to predict the temperatures and pressures necessary for this conversion. Calculations based on values of these properties indicate that it should be possible to form diamonds from graphite at temperatures between 1800 and 2300 °K and at pressures between 55,000 and 75,000 atmospheres. Synthetic diamonds have been made under these conditions.

The apparatus consists of a vertical metal tube, the upper end of

which is surrounded by an electric furnace and the lower end of which extends into the Bunsen ice calorimeter. Around the lower end of the tube is formed a solid mantle of ice, which in turn is surrounded by ice water. In the bottom of the calorimeter is a pool of mercury, which is connected to an outside reservoir through a small tube. The water-ice-mercury vessel is sealed from the atmosphere, and is insulated by a surrounding bath maintained at the ice point.

Heat capacity measurements were made on 20.264 grams of polished gem diamonds, having an impurity content of less than 0.2 weight per cent. The diamonds, after thorough cleaning, were placed in a cylindrical silver capsule having an internal volume of about 8 cm^3. To prevent oxidation of the diamonds at elevated temperatures, air was removed from the capsule by prolonged pumping, helium at 200-mm mercury pressure was introduced, and the cylinder was sealed. A second silver capsule, used as a blank, was fabricated so as to resemble closely that containing the diamonds.

Measurements were made by first suspending the sample container within the furnace on a wire-pulley arrangement. The temperature of the sample container in the furnace was determined with a platinum resistance thermometer below 873°K, and with a thermocouple above this temperature.

After heating, the sample was dropped (nearly free fall) into the ice calorimeter, where it melted some ice in cooling. The resulting volume decrease (the water formed is more dense than the ice) drew mercury into the system, the mass of mercury being proportional to the heat evolved by the sample in cooling. By carefully weighing the mercury reservoir before and after each run, the mass of the mercury was determined. The heat equivalence of one gram of mercury has been established through accurate calibration of the apparatus.

A pneumatic braking device attached to the suspension wire prevented impact of the container with the calorimeter. A gate within the tube was opened only during the drop to prevent transfer of radiation down from the furnace. Circular platinum shields just above the container reduced errors due to heat losses during cooling. The tube through which the sample was dropped was helium-filled to reduce the time required to reach equilibrium in both the furnace and calorimeter.

Determinations were made at 50-degree intervals from 323 to 673 °K, and at 100-degree intervals from 673 to 1073 °K. Replicate runs were made at each temperature except the highest (1073 °K, at which point a noticeable leak developed in the container), each run having a different heating time to reveal any time-dependence of the data.

Measurements were also made at each temperature with the empty container. These blank determinations provided a measurement of the quantity of heat contributed by the sample container. The heat contributed by the diamonds alone was found by subtracting the value of the container from the total heat measured.

BOOK REVIEWS

SURFACE ACTIVITY AND DETERGENCY, edited by K. Durham. 236 pages, diagrams, plates, 6 × 9 in. London, MacMillan & Co., Ltd.; New York, St. Martin's Press; 1961. Price, $8.00.

The authors have presented a well-written account of the basic physical chemistry of detergent systems and examined detergent action from a fundamental standpoint. The authors have obtained their material from a critical review of the literature with interjection of some of their own research and ideas on the subject.

The book starts with a discussion of the properties of detergents and their solutions, and then applies the ideas developed here to the fundamental processes of detergent action—that is, wetting, removal of dirt, and the prevention of redeposition. In dealing with detergent action the authors have considered more critically than others the nature of the dirt and its interaction with the components of the detergent solution. This has led to a new plausible explanation of the building action of condensed phosphates. Kinetic effects in detergency, which have not been covered in other works, are discussed.

In each chapter the authors have concentrated on an explanation of the basic theories involved, then presented data supporting these theories. As theories are logically developed for the properties of detergents and then applied to the over-all theory of detergent action, the authors gradually introduce more information on the applications of this knowledge to practical problems. Detergency evaluation is covered but briefly due to other excellent works on this subject.

For those interested only in the practical aspects of detergency, this book is not recommended. It gives, however, an excellent survey of all phases of detergency theory for those whose interest lies in this area. Although some may desire a more thorough treatment of specific subjects, I think the authors used good judgment in restricting depth of coverage within reasonable grounds.

R. C. TAYLOR
The Atlantic Refining Company

THE PHYSICAL PRINCIPLES OF ASTRONAUTICS, by Arthur I. Berman. 350 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $9.25.

Occasionally, there appears a book in which a brilliant teacher distills all his knowledge to produce a superb end product. In most cases professors who write books do so from accumulated lecture notes. In many cases the organization of the notes is faulty which results in a book in which some of the necessary elements are lacking. Thus, while professors can and do write books they cannot all be classed as acceptable.

A notable exception to this is *The Physical Principles of Astronautics* written by Dr. Arthur I. Berman, Associate Professor of Physics at Rensselaer Polytechnic Institute. In this book of three sections the author has covered the field of dynamical astronomy and space flight. While the author indicates the book can be used as a textbook in a one-semester course at the undergraduate level, this reviewer believes that unless you are dealing with exceptional students the book is designed for the beginning graduate level. However, this is not to be construed as a criticism of the book. Some instructors make their students "reach" and this book is eminently qualified on that score.

Part I discusses the principles of astronomy. The first chapter is devoted to the physical characteristics of the earth and the moon while the second chapter is involved with the physics of the solar system.

Part II, Foundations of Mechanics, discusses the fundamental concepts of mechanics, such as weight, mass, potential energy, and the Coriolis force. In the two chapters comprising this section the author has done a creditable job in presenting some difficult concepts. The table showing the relationship between absolute and gravitational units is excellent.

The last part is devoted to the dynamics of space flight. Here are developed the properties of orbits and transfer trajectories. Here the effects of perturbative forces are described.

The author has presented illustrative prob-

lems within the body of the book and at the conclusion of each chapter are found series of problems some of which are presented with their answers.

There are eight appendices, a glossary and a fine bibliography incorporated in the book. The appendices are most helpful and pertinent to the subject. While these could have been incorporated in the body of the book their appearance as appendices helps preserve the theme the author tries to develop. While some will find the appendices of great help, others will find them elementary.

The Physical Principles of Astronautics is a wonderful addition to the literature and should find its way into our colleges and universities as the demand for the teaching of astronautics increases.

I. M. LEVITT
The Fels Planetarium

PROCEEDINGS OF AN INTERNATIONAL CONFERENCE ON INSTRUMENTATION FOR HIGH-ENERGY PHYSICS, edited by C. E. Mauk, A. H. Rosenfeld and R. K. Wakerling. 321 pages, diagrams, illustrations, 8½ × 10½ in. New York, Interscience Publishers, 1961. Price, $10.00.

This volume constitutes the published proceedings of an international conference held at the E. O. Lawrence Radiation Laboratory, Berkeley, California, on September 12, 13, and 14, 1960. It would appear to be of interest mainly to would-be specialists (that is, graduate students) and specialists, although experimentalists in other branches of physics will find material of interest. The topics covered include beam transport and analysis (including beams of such secondary particles as mesons and high energy neutrinos), the production of high magnetic fields (100–200,000 gauss), electronic circuits, detectors (especially Cerenkov and spark counters), bubble chambers, and the reduction of data from bubble-chamber film. This last topic was possibly the most exciting one of the conference. A bubble-chamber picture may contain a considerable amount of information but extraction of this information and its reduction to a simple statement of the physics involved may be quite tedious for a human operator using only such simple tools as curvature templates and desk calculators. Such procedures are impossibly slow for existing high-energy accelerators, which produce significant pictures as fast as several per minute. The semi-automatic procedures in use at the time of this conference involved several skilled humans for scanning, supervision, and checking, some special "measuring machines," and one or two high-speed computors (for example, IBM 704's). The rate at which "events" in the bubble-chamber pictures could be analyzed depended on the complexity of the events, but was typically a hundred or so per day. The clearly necessary fully automatic systems were said to be several years in the future. The apparent simplicity of some of the problems that machines have not yet been taught to solve offers comfort to mere humans, and some assurance that they will not be outmoded in the near future. For example, machines could not (at the time of the conference) look at a bubble-chamber picture and separate significant tracks (a series of bubbles lying on a straight or slowly curving line) from random single bubbles, low-energy electron spirals, and other background "hash."

Several adverse comments could be made about this book and its contents, but perhaps it is more charitable to observe that, for archival purposes at least, the proceedings of an international conference should be published essentially in full.

V. K. RASMUSSEN
Bartol Research Foundation

MICROWAVE FERRITES, by P. J. B. Clarricoats. 253 pages, diagrams, 6 × 10 in. New York, John Wiley & Sons, Inc., 1961. Price, $8.00.

This book is primarily intended for the microwave engineer who desires some fundamental understanding of solid state microwave devices employing materials which are collectively classified as ferrites. The introductory chapter titled "General Properties of Ferrites" is concerned with magnetic and dielectric properties of ferrites. As such it forms a good description of the fundamental concepts used by the magnetic specialist. It is interesting for a physicist to note that in this introductory chapter an engineer's love of the M.K.S. system of units does not outweigh the convenience of using the C.G.S. system when describing atomic magnetism. The bulk of the text is concerned with the propagation of electromagnetic waves in ferrites. This is discussed in a fundamental way with little

regard to practical device operation but rather emphasizes the operation of devices in terms of the phenomenological description outlined in the earlier chapters. F. T. HEDGCOCK

The Franklin Institute Laboratories

METEOR SCIENCE AND ENGINEERING, by D. W. R. McKinley. 309 pages, diagrams, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $12.50.

There has occurred a spectacular crossing of scientific disciplines in the past two decades. Due to the heavy subsidizing of science in both universities and industry many teachers and students have willingly and unwillingly, entered into new and foreign fields. An example of this crossover is in the application of radio techniques to astronomy.

Following the end of World War II a large number of radar dishes were suddenly made available to astronomers. This reviewer remembers the Giacobini meteor shower of October 1946 in which radio techniques were used to count the unusually large number of meteors entering the earth's atmosphere. Following this significant event astronomers began applying this new tool to many problems in astronomy. By the same token engineers began fashioning new tools and using them to investigate astronomical problems. This marriage of sciences has resulted in a tremendous advance in meteor astronomy.

Meteor Science and Engineering is a book written by a scientist and engineer on a highly specialized astronomical subject and the result is a thoroughly authoritative work meriting wide professional acceptance. While Dr. McKinley is at present Associate Director of the Radio and Electrical Engineering Division of the National Research Council of Canada, he has been involved in meteor research for the past 14 years. The author in his preface acknowledges his debt to meteor specialists like Whipple, Millman, Hawkins, McCrosky and others. The material the author has put together illustrates the professional touch of the authorities on whose work he has drawn. An example of this touch may be seen in the most modern of the theories which are proposed to account for the origin of meteors, their distribution, the question of whether they are members of the solar system or whether some are intruders from beyond the solar system. The author does not pretend he knows the answers. He pointedly discloses that we lack acceptable solutions to many of the problems posed in this field.

The book arrangement is a tribute to the skill of the author. He begins with a historical survey of the meteor problem and shows that back in the mid-1920s scientists in England (Appleton and Barnett) and the United States (Breit and Tuve) laid the groundwork for future radio studies of meteors by discovering the reflection of radio waves from the ionospheric layers. He then describes the researches undertaken in the 1930's which set the stage for the development of the science after World War II.

In Chapter 2 the author discusses elementary astronomy and radio principles to provide a background for what follows. His Chapter 3 describes visual and photographic techniques for observing meteors to derive significant data from them. The detectors range from the naked eye to binoculars to Schmidt cameras to photoelectric cells. Chapter 4 concerns radio techniques for meteor detection.

Chapters 5 and 6 are the most impressive ones in the book. The first deals with observational data on meteors and the second on astronomical aspects of meteors. These two chapters can be considered classics in this field. In the first of the chapters the author has compressed a wealth of material to survey the field. Dr. McKinley treats almost every aspect of meteor observations including those which appear manifestly in error. In the later chapter he discusses meteor streams and cometary associations and the diagrams of the shower meteors are a splendid addition to the book.

In Chapter 7 the author discusses the physical theory of meteors. Here are treated the luminosity and the ionization of meteors entering the atmosphere. It is admitted that the treatment in this chapter is superficial but the subject is so nebulous that it is impossible to provide a definitive treatment.

The remaining three chapters deal with some aspects of radio-meteor effects.

Meteor Science and Engineering is an outoutstanding book on meteoritics. It is a reference *par excellence*. I. M. LEVITT

The Fels Planetarium

BOOK NOTES

THE SCIENCE OF ADHESIVE JOINTS, by J. J. Bikerman. 258 pages, diagrams, 5¼ × 8 in. New York, Academic Press Inc., 1961. Price, $8.00.

In this monograph, the author stresses the science of adhesive joints (which he calls *adhints*), rather than the formulation of specific adhesives. After an introductory chapter explaining the terms used in bonding, the author defines adhesive joints and devotes one chapter to the formation of such joints including wetting, measurement of contact angles, and pretreatment of adherent surfaces. Four chapters cover, respectively, tack, setting, final strength of adhints, and stresses in adhints. Experimental strength of adhints is dealt with in Chapter 8; Chapter 9 describes a few of the less common tests employed in adjudging adhints. The final chapter is a short summary for the "practical" man seeking to find out what is wrong with a weak joint.

PROGRESS IN CRYOGENICS, edited by K. Mendelssohn. Volume 3, 173 pages, diagrams, plates, 6 × 9¼ in. New York, Academic Press Inc., 1961. Price, $8.00.

Volume 3 of this series contains the following papers: "Helium Liquefiers," by A. J. Croft; "Low Temperature Heat Exchangers," by A. G. Lenfestey; "Novel Refrigeration Cycles and Devices," by W. E. Gifford; "Cryogenic Rocket Propellants," by I. E. Smith; "Paramagnetic Substances for Nuclear Orientation," by R. P. Hudson; and "Dynamic Nuclear Orientation," by C. D. Jeffries.

DYNAMICS: VOLUME 2 OF ANALYTICAL ELEMENTS OF MECHANICS, by Thomas R. Kane. 324 pages, diagrams, 5¼ × 8¼ in. New York, Academic Press Inc., 1961. Price, $6.25.

Volume 2 of this undergraduate text in classical mechanics deals with dynamics. Stressing kinematics and the theory of moments and products of inertia, the author begins his discussion with a chapter on the differentiation of vectors. Chapter 2, on kinematics, deals with rates of change of orientation, angular velocity and acceleration, relative velocity and acceleration, and absolute velocity and acceleration. Chapter 3 discusses second moments of a point, a set of points, curves, surfaces and solids, and sets of particles. The fourth and final chapter is on the laws of motion. Twelve sets of problems, with answers, appear at the end of the book.

PLASTICS IN NUCLEAR ENGINEERING, by James O. Turner. 134 pages, illustrations, 5 × 7½ in. New York, Reinhold Publishing Corp., 1961. Price, $5.50.

This book describes the ways in which plastics are made to serve the nuclear scientist and engineer. It is intended for designers and operating personnel concerned with Bevatrons, reactors, bubble chambers and other equipment used in nuclear engineering. The book covers applications of plastics to the measurement of radiation, radiation protection, high-voltage service, low-voltage service, magnets, high-vacuum apparatus, and optical apparatus. Mechanical and thermal uses of plastics are dealt with, and a final chapter covers effects of radiation on plastics.

FLUID MECHANICS, by Richard H. F. Pao. 484 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $7.50.

An introductory text in fluid mechanics, this new work emphasizes the concepts and principles of fluid motion that are common to applications in all branches of engineering. The author stresses physical concepts and uses rational approaches in developing the fundamental equations. After an introductory chapter, ten chapters cover: fluid statics, fluid kinematics, fluid dynamics, fluid viscosity and flow of real fluids, dimensional analysis and model similitude, flow of incompressible fluids in closed conduits, fluid compressibility and compressible flow, fluid flow about immersed bodies, dynamic lift, and flow of liquids in open channels.

ADVANCES IN GEOPHYSICS, VOLUME 7, edited by H. E. Landsberg and J. Van Mieghem. 321 pages, diagrams, illustrations, 6 × 9 in. New York, Academic Press Inc., 1961. Price, $11.00.

Volume 7 of this series presents six articles dealing with various phases of geophysical research: "Developments in Controlled Experiments on Larger Scale Geophysical Problems," by Dave Fultz; "Atmospheric Tides," by Manfred Siebert; "Generalized Harmonic Analysis," by J. Van Isacker; "Temperature and Wind in the Lower Stratosphere," by H. A. Panofsky; "Arctic Meteorology (A Ten-Year Review)," by A. D. Belmont; and "Phase Relations of Some Rocks and Minerals at High Temperatures and High Pressures," by George C. Kennedy.

DISPERSION RELATIONS, edited by G. R. Screaton. 290 pages, diagrams, 6 × 10 in. New York, Interscience Publishers Inc., 1961. Price, $9.25.

The eight papers in this volume were originally delivered as lectures at the first Scottish Universities Summer School, held in 1960. The papers, on a very advanced level, are: "Introduction to Dispersion Relation Techniques," by J. D. Jackson, "The Analytic Properties of Perturbation Theory," by J. C. Polkinghorne; "Proof of Some of the Analytic Properties of the Relativistic Scattering Amplitude," by W. Thirring; "Practical Utilization of the Nearest Singularity in Dispersion Relations," by M. J. Moravcsik; "Double Dispersion Relations and Unitarity as the Basis for a Dynamical Theory of Strong Interactions," by G. F. Chew; "The Electromagnetic Structure of Pions and Nucleons," by W. R. Frazer; "The Use of One-Dimensional Representations in Pion Physics," by S. Fubini; and "The Existence of the Hamiltonian for Casual Systems," by J. M. Jauch.

BOUNDARY LAYER AND FLOW CONTROL, VOLUME 1, edited by G. V. Lachmann. 600 pages, diagrams, illustrations, 6 × 9 in. Oxford, Pergamon Press Ltd., 1961. Price, $35 for set of two volumes.

Volume 1 of this two-volume work presents, in part I, a very comprehensive history of boundary layer and flow control research in Germany, France, Great Britain and the United States. Part II, consisting of sixteen papers, deals with boundary layer and flow control to prevent separation and/or increase lift in subsonic flow. Emphasis is on engineering applications, but fundamental concepts, theory, experimental methods, and design are also included. The 38 contributors are all recognized authorities, and much of the material represents new contributions not previously published (for example: calculation of sucked boundary layers in two- and three-dimensional flow; research on suction surfaces; design of low-drag aircraft; and control of boundary layer separation in the presence of shock waves).

MECHANICAL BEHAVIOR OF MATERIALS AT ELEVATED TEMPERATURES, edited by John E. Dorn. 514 pages, diagrams, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $14.50.

This versatile work can serve either as the sole text for senior or first-year graduate courses or it may be used as a supplementary text or reference work (in Mechanical Behavior of Materials, Dislocation Theory, Mechanical Metallurgy, and Fatigue and Fractures). The major aim is to give the scientist and engineer a systematic background in dislocation theory and its applications to creep and related high-temperature properties. Twelve authors have contributed fifteen papers covering all phases of the subject.

COMBUSTION AND PROPULSION, edited by A. L. Jaumotte, A. J. Lefebvre and A. M. Rothrock. 396 pages, diagrams, illustrations, 6 × 10 in. Oxford, Pergamon Press Ltd., 1961. Price, $15.00.

These proceedings of the Fourth AGARD Colloquium on High Mach Number Air-Breathing Engines are divided into seven parts, covering different aspects of the subject: I, The Future of Air-Breathing Engines (2 papers); II, Performance and Application (2 papers); III, Hypersonic Inlet Studies (2 papers); IV, Diffusion Flames and Detonation Waves (2 papers); V, Nozzle Flow with Chemical Reactions (2 papers); VI, Research in Turbomachinery (1 paper); and VII, High Temperature Material Problems (2 papers). Discussions (some of which are in French) and authors' conclusions add to the value of this work.

THE EARTH TODAY, edited by A. H. Cook and T. F. Gaskell. 404 pages, diagrams, 6 × 10 in. New York, Interscience Publishers, Inc., 1961. Price, $13.00.

This is a bound edition of a special issue of the *Geophysical Journal*, published in honor of Sir Harold Jeffreys. The 27 papers comprising the issue represent the most advanced work in a number of fields in which Sir Harold has been interested. Among the 30 contributors are many of Sir Harold's colleagues and former students. Included among the subjects covered are the Earth's gravitational field, seismology, free oscillations, atmosphere and oceans, thermal state, and instruments. The book should be of interest to astronomers, physicists, geologists, geophysicists, and mathematicians.

ISOPROPYL ALCOHOL, by Lewis F. Hatch. 184 pages, diagrams, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $7.00.

Anyone who is responsible for preparing, specifying or testing isopropyl alcohol will be interested in this book. It is a general survey of the history, production, properties, and uses of isopropyl alcohol, with special emphasis on the pertinent patent literature and on the applications. Useful tables of data cover uses, properties, and applications.

TRANSISTORS AND ACTIVE CIRCUITS, by John G. Linvill and James F. Gibbons. 501 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $14.50.

This text discusses fundamental problems in circuits whose active elements are transistors. The material is divided into three parts: 1, The Physics of Semiconductors (8 chapters); 2, Two-Port Network Theory (7 chapters); and 3, Transistor Circuits (7 chapters). In Part 1, the development of physical models is based on a classical model of carrier motion in a solid, not on the usual quantum theory. This necessitates omission of the study of devices (such as the tunnel diode) in which quantum-mechanical effects play an important part. There are other interesting departures from the usual approach. Problems are provided for most chapters. The text is suitable for graduate students and practicing engineers.

MINIATURIZATION, edited by H. D. Gilbert. 296 pages, diagrams, illustrations, 6 × 9 in. New York, Reinhold Publishing Corp., 1961. Price, $10.00.

Twenty experts present sixteen papers in particular phases of miniaturization. With today's emphasis on space-saving, this valuable compilation covers military equipment, aircraft electronic devices, missiles and satellites, medicine (for example, miniature electrodes for biological recordings), communications, computers, consumer products (hearing aids, etc.), and space travel. In addition, there are chapters covering the problems of tooling up to produce miniaturized products, the design, manufacture and maintenance of tiny devices. A final chapter looks ahead at rather staggering possibilities for this branch of technology—one such is the reproduction of all existing books on a space the size of a million pinheads, an area of about 3 square yards!

PROCEEDINGS OF THE SECOND INTERNATIONAL CONFERENCE ON OPERATIONAL RESEARCH, edited by J. Banbury and J. Maitland. 810 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $15.00.

This volume records the proceedings of the Second International Conference on Operational Research, held in September, 1960, in France. Eighty-five papers, divided into sixteen topics, are given in the book; some are printed in full in English, with a French resume; the others vice versa. Roughly half the papers are of a general nature; the others are on applications in particular fields.

CURRENT TRENDS IN SCIENTIFIC RESEARCH, by Pierre Auger. 229 pages, 8½ × 11 in. New York, Columbia University Press (International Documents Service), 1961. Price, $6.75.

Prepared under the auspices of the United Nations, this survey covers the main trends of inquiry in the field of the natural sciences and the trends affecting the organization of scientific research and the dissemination of results. It also offers recommendations concerning the application of scientific knowledge for peaceful ends. Many fields are treated, including theoretical, nuclear and atomic physics, earth

and space sciences, medical sciences, food and agricultural sciences, fuel and power research, and industrial research. Among the many recommendations in the survey is the creation of an international service (for UN member nations) capable of advising governments in their efforts to improve national scientific research.

PUBLICATIONS RECEIVED

COMBUSTION, FLAMES, AND EXPLOSIONS OF GASES, by Bernard Lewis and Guenther von Elbe. Second edition, 731 pages, diagrams, illustrations, 6 × 9 in. New York, Academic Press Inc., 1961. Price, $22.00.

DESIGN OF MACHINE ELEMENTS, by M. F. Spotts. Third edition, 583 pages, diagrams, 6 × 9 in. Englewood Cliffs (N. J.), Prentice-Hall, Inc., 1961. Price: $13.35 (trade); $10.00 (text).

LINEAR ALGEBRA AND GROUP THEORY, by V. I. Smirnov, revised, adapted and edited by Richard A. Silverman. 464 pages, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $12.50.

INTRODUCTION TO VECTOR ANALYSIS, by Harry F. Davis. 349 pages, diagrams, 6 × 9 in. Boston, Allyn and Bacon, Inc., 1961. Price, $7.95.

STABILITY IN NONLINEAR CONTROL SYSTEMS, by A. M. Letov, translated by J. George Adashko. 312 pages, diagrams, 6 × 9 in. Princeton, Princeton University Press, 1961. Price, $8.50.

ELECTRICAL CONTRACTING, by Ray Ashley. 294 pages, diagrams, illustrations, 7½ × 10 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $10.00.

FUNCTIONAL TRIGONOMETRY, by Abraham P. Hillman and Gerald L. Alexanderson. 322 pages, diagrams, 6 × 9 in. Boston, Allyn and Bacon, Inc., 1961. Price, $5.95.

ELEMENTARY DIFFERENTIAL EQUATIONS, by William Ted Martin and Erie Reissner. Second edition, 331 pages, 6 × 9 in. Reading (Mass.), Addison-Wesley Publishing Co., Inc., 1961. Price, $6.75.

THE FRANKLIN INSTITUTE

MINUTES OF THE STATED MEETING

October 18, 1961

The Stated Meeting of The Franklin Institute was held at 8:15 P.M. in Franklin Hall, following the Annual Medal Day Reception and Dinner at which approximately 375 members and guests were present.

After introducing Former Medalists who were present, and paying tribute to the Committee on Science and the Arts for their unselfish service to The Institute and to the scientific world, Wynn Laurence LePage, President, called the meeting to order and announced that the minutes of the May 1961 Stated Meeting had been printed in the June issue of the JOURNAL. There being no corrections or additions, the minutes were approved as published.

Mr. LePage then announced that in accordance with tradition, the Board of Managers had nominated the 1961 Franklin Medalist, Dr. Detlev W. Bronk, for Honorary Membership in The Franklin Institute. Upon approval by the membership, Mr. LePage presented Dr. Bronk with the Certificate of Honorary Membership and membership card.

Mr. LePage then spoke of the specific steps The Institute is taking to expand and improve our services to the community and the country through our Program to Advance Science Education, announced earlier this year. This ten-year program described by Dr. James R. Killian, Jr., President Eisenhower's Science Advisor, as "bold and well conceived" is receiving enthusiastic and generous support of industry, foundations and members and friends. The president then read the following telegram from the President of the United States, John F. Kennedy:

> "To all assembled for The Franklin Institute Medal Day, I wish to send greetings and warm congratulations to those being honored for their achievements in Science and the Mechanic Arts.
>
> "Benjamin Franklin knew that progress in science and technology was greatly enhanced by the free and open exchange of knowledge and ideas. He sought to encourage the true universality of science and to tear asunder any barriers to free communication of truth. The Franklin Institute honors the wisdom of Benjamin Franklin by recognizing important contributions to science and technology from all corners of the world. Today, the Institute also pays tribute to those whose discoveries and achievements will redound to the benefit of all men everywhere. On behalf of my countrymen, I am pleased to extend this nation's appreciation and gratitude for those advances in new knowledge which yield their return to the progress of civilization.
>
> JOHN F. KENNEDY"

The President then recognized the Sponsor for each Medalist, each Sponsor presenting his candidate for an Award. The presentation of the Medals was made by President LePage.

Following the presentation of the Medals, the President introduced the speaker of the evening, Dr. Detlev W. Bronk, the Franklin Medalist for 1961, President of The Rockefeller Institute, New York, N. Y. Dr. Bronk's outstanding talk on "The Humane Qualities of Science" was received with unanimous enthusiasm.

The President expressed warm and sincere thanks to Dr. Bronk for his fascinating talk, and thanked all present for a full and enjoyable evening.

The meeting was adjourned at 9:40 P.M.

WILLIAM F. JACKSON, JR.
Secretary

NOTE: The entire proceedings of Medal Day will be published in the December issue of the JOURNAL.

1961 MEDALS

Cresson Medals (1848)

DONALD A. GLASER
Professor of Physics
University of California
Berkeley, California

REINHOLD RÜDENBERG
Gordon McKay Professor of Electrical Engineering Emeritus
Harvard University
Cambridge, Massachusetts

RUDOLF L. MÖSSBAUER
Senior Research Fellow in Physics
California Institute of Technology
Pasadena, California

JAMES ALFRED VAN ALLEN
Head, Department of Physics and Astronomy
State University of Iowa
Iowa City, Iowa

Longstreth Medals (1890)

ROBERT L. ALCORN, JR.
Research Engineer
Chambersburg Engineering Company
Chambersburg, Pennsylvania

EUGENE C. CLARKE
Formerly President
Chambersburg Engineering Company
Chambersburg, Pennsylvania

HENRY A. WEYER
Chief Engineer
Chambersburg Engineering Company
Chambersburg, Pennsylvania

JOSIAH L. MERRILL, JR
Member of the Technical Staff
Bell Telephone Laboratories, Inc.
Murray Hill, New Jersey

Levy Medal (1923)

JOHN H. HOLLAND
Assistant Professor of Communication Sciences
The University of Michigan
Ann Arbor, Mich.

Wetherill Medals (1925)

ALBERT E. HITCHCOCK
Boyce Thompson Institute for Plant Research
Yonkers, New York

PERCY W. ZIMMERMAN
Boyce Thompson Institute for Plant Research
Yonkers, New York

(*Awarded Posthumously*)

Brown Medal (1938)

CHARLES-EDOUARD LE CORBUSIER
Architect
Paris, France

Clamer Medal (1943)

RUSSELL P. HEUER
Vice President
General Refractories Company
Philadelphia, Pennsylvania

Ballantine Medals (1946)

NICOLAAS BLOEMBERGEN
Gordon McKay Professor of Applied Physics
Harvard University
Cambridge, Massachusetts

H. E. DERRICK SCOVIL
Head, Solid State Maser Department
Bell Telephone Laboratories, Inc.
Murray Hill, N. J.

LEO ESAKI
Senior Physicist
International Business Machines Research Center
Yorktown Heights, New York

Franklin Medal (1914)

DETLEV W. BRONK
President
The Rockefeller Institute
New York, New York

NEW BOOKS IN THE FRANKLIN INSTITUTE LIBRARY

PHYSICS

Atomic Energy Levels in Crystals, by J. L. Prather. U. S. Department of Commerce, National Bureau of Standards. 1961.

Theory of Elastic Stability, by S. Timoshenko. 2nd ed. McGraw-Hill. 1961.

Relations de Dispersion et Particules Élémentaires, by C. DeWitt. Wiley, 1960.

Handbook of Fluid Dynamics, edited by V. L. Streeter. McGraw-Hill. 1961.

Kernenergie-Technik, by H. Engel. Verlag Moderne Industrie. 1960.

Power Reactor Technology, by J. K. Pickard. Van Nostrand. 1961.

The Philosophical Impact of Contemporary Physics, by M. Capek. Van Nostrand. 1961.

Plasmas and Controlled Fusion, by D. J. Rose. M.I.T. Press. 1961.

Conference on Very High Pressure. Proceedings, edited by F. P. Bundy. Wiley. 1961.

An Introduction to Relativistic Quantum Field Theory, by S. S. Schweber. Row, Peterson. 1961.

The Physics of Radiology, by H. E. Johns. 2nd ed. Thomas. 1961.

A Spectrophotometric Atlas of the Spectrum of CH from 3000Å to 5000Å. National Bureau of Standards. 1961.

Bibliography of Temperature Measurement, Jan. 1953 to June 1960, by C. Halpern. National Bureau of Standards. 1961.

Thermoelectricity, by R. Heikes. Interscience. 1961.

CHEMISTRY AND CHEMICAL ENGINEERING

Asphalts and Allied Substances, by H. Abraham. 6th ed. Van Nostrand. 1960.

Azo and Diazo Chemistry, by H. Zollinger. Interscience. 1961.

Abundance of Chemical Elements, by V. V. Cherdyntsev. University of Chicago Press. 1961.

Chemical Crystallography, by C. W. Bunn. 2nd ed. Clarendon Press. 1961.

Radiation Chemistry of Gases, by S. C. Lind with C. J. Hochanadel and J.A.Ghormley. Reinhold. 1961.

Bibliography of Glass, by G. S. Duncan. Dawsons. 1960.

The Chemistry and Mode of Action of Herbicides, by A. S. Crafts. Interscience. 1961.

Isopropyl Alcohol, by L. F. Hatch. McGraw-Hill. 1961.

Black Gold, by A. B. Thompson. Doubleday. 1961.

Rare Earth Alloys, by K. A. Gschneidner. Van Nostrand. 1961.

X-Ray Analysis of Organic Structures, by S. C. Nyburg. Academic Press. 1961.

CIVIL ENGINEERING

Ausfluss, Uberfall und Durchfluss im Wasserbau, by F. H. Knapp. G. Braun. 1960.

Bibliography of Hydrometry, by S. Kolupaila. University of Notre Dame Press. 1961.

Pile Foundations, by R. D. Chellis. 2nd ed. McGraw-Hill. 1961.

Papers on Road and Paving Materials (Bituminous). American Society for Testing Materials. 1961.

Capacities of Stacks in Sanitary Drainage Systems for Buildings, by R. S. Wyly. National Bureau of Standards. 1961.

Symposium on Nuclear Methods for Measuring Soil Density and Moisture. American Society for Testing Materials. 1961.

Statik der Pfalwerke, by F. Schiel. Springer. 1960.

Water Treatment for Industrial and Other Uses, by E. Nordell. 2nd ed. Reinhold. 1961.

ELECTRICAL ENGINEERING

Electromechanical System Theory, by H. E. Koenig. McGraw-Hill. 1961.

Promyshlennaia Telemekhanika. Akademii nauk SSSR. 1960.

ERROR-CORRECTING CODES, by W. W. Peterson. M.I.T. Press. 1961.
OPERATIONAL ELECTRICITY, by C. I. Hubert. Wiley. 1961.
STATIC RELAYS FOR ELECTRONIC CIRCUITS, edited by R. F. Blake. Reinhold. 1961.
ANALOG AND DIGITAL COMPUTER TECHNOLOGY, by N. R. Scott. McGraw-Hill. 1960.
MATHEMATICAL MACHINES, by F. J. Murray. Columbia University Press. 1961.
DIGITAL COMPUTER FUNDAMENTALS, by T. C. Bartee. McGraw-Hill. 1960.
TENSOR ANALYSIS OF ELECTRIC CIRCUITS AND MACHINES, by L. V. Bewley. Ronald Press. 1961.

SPACE SCIENCE

BOUNDARY LAYER AND FLOW CONTROL, by G. V. Lachman. Pergamon. 1961.
NEW METHODS IN LAMINAR BOUNDARY-LAYER THEORY, by D. Meksyn. Pergamon. 1961.
HUMAN FACTORS IN JET AND SPACE TRAVEL, edited by S. B. Sells. Ronald Press. 1961.
PRINCIPLES OF INERTIAL NAVIGATION, by C. J. Savant. McGraw-Hill. 1961.
SOLID PROPELLANT ROCKET RESEARCH, edited by Martin Summerfield. Academic. 1960.
THE THEORY OF SUBSONIC PLANE FLOW, by L. C. Woods. Cambridge University Press. 1961.

MECHANICAL ENGINEERING

INTRODUCTION TO ENGINEERING MECHANICS, by J. V. Huddleston. Addison-Wesley. 1961.
OPTIMUM DESIGN OF MECHANICAL ELEMENTS, by R. C. Johnson. Wiley. 1961.
MECHANICAL BEHAVIOR OF MATERIALS AT ELEVATED TEMPERATURES, by J. E. Dorn. McGraw-Hill. 1961.
PROGRESS IN REFRIGERATION SCIENCE AND TECHNOLOGY. International Congress of Refrigeration. Pergamon. 1960.
DEVELOPMENT OF HIGH TEMPERATURE STRAIN GAGES, by J. W. Pitts. National Bureau of Standards. 1961.
AERO-THERMODYNAMICS AND FLOW IN TURBOMACHINES, by M. H. Vavra. Wiley. 1960.

MATHEMATICS

A GUIDE TO MATHEMATICAL TABLES, by A. V. Lebedev. Pergamon. 1960.
MATHEMATICS IN THE MAKING, by L. T. Hogben. Macdonald. 1960.
THE ALGEBRA OF PROBABLE INFERENCE, by R. T. Cox. Johns Hopkins. 1961.

COMMITTEE ON SCIENCE AND THE ARTS

(Abstract of Proceedings of Stated Meeting held Wednesday, October 11, 1961.)

HALL OF THE COMMITTEE,
PHILADELPHIA, OCTOBER 11, 1961.

MR. JOSEPH GRAY JACKSON in the Chair.

The following report was presented for final action:

No. 3386: Drilled-in Caissons.

This report recommended the award of a Certificate of Merit to Joseph H. Thornley, of New York, New York, "For his invention and development of drilled-in caissons, whose economy, simplicity, ease of construction and high load-carrying capacity have substantially advanced the engineering technology of foundation construction."

D. S. FAHRNEY,
Secretary to Committee

CURRENT TOPICS

Semiconducting Diamonds.—Methods have been discovered at the General Electric Research Laboratory which make it possible for the first time to produce semiconducting diamonds, according to Dr. Guy Suits, vice president and director of research. Such diamonds are extremely rare in nature, accounting for less than one per cent of natural diamonds, but can now be grown at will in the laboratory using the high-temperature, ultra-high pressure process first developed at the Laboratory.

Semiconducting borazon has also been made successfully, Suits announced. Borazon, which is a cubic form of boron nitride, has a structure very similar to that of diamond and is equally hard. It was first made at the General Electric Research Laboratory.

The method for making semiconducting diamonds was discovered by Dr. Robert H. Wentorf, Jr., of the Research Laboratory and Harold P. Bovenkerk, of GE's Metallurgical Products Department. Semiconducting borazon was discovered by Dr. Wentorf, who also developed the original process for making borazon. Diamonds are made semiconducting by adding impurities such as boron, beryllium or aluminum to the mixture of graphite and catalyst from which diamonds are made. The mixture is subjected to pressures of about 1 million pounds per square inch and temperatures above 2000 F. Under these conditions, diamonds form with concentrations of one per cent or less of the desired impurity, and have electrical conductivities in the semiconducting range (intermediate between those of metals and insulators).

Drs. Wentorf and Peter Cannon of the Research Laboratory have also prepared semiconducting diamonds by diffusing boron and aluminum into Man-Made or natural diamonds at high pressures and temperatures. All the semiconducting diamonds made so far have been *p*-type (positive current carriers). Both *p*-type and *n* (negative current carrier) -type crystals are necessary in transistors and other semiconducting devices, and a search for processes that will produce *n*-type diamonds is continuing.

In borazon, both *p*-type and *n*-type crystals can be produced, and one type can be grown onto a "seed" of the other type to form *p-n* junctions. Tests on such junctions have shown that they act as rectifiers (allow current to flow in only one direction). Beryllium as an impurity produces *p*-type borazon, and a number of substances including sulfur, silicon, many organic compounds, and potassium cyanide, when added to the synthesis mixture, result in *n*-type borazon.

The semiconducting diamonds prepared with boron are blue, in shades ranging from a pale blue-white to a deep blue-black, depending on how much boron is present in the crystal. Semiconducting diamonds found in nature, which have been studied for many years by a number of investigators, are also sometimes blue. One of the most famous blue-white diamonds is the Hope diamond, and although its conductivity has not been measured, its color suggests that it is probably a semiconductor.

Teflon-Coated Metals and Laminates.—Two new types of insulating and printed circuit materials have

been announced by Amflex Products Department of American Machine & Foundry Company. Both types are called AMFOIL and include Teflon-coated foils and laminates of metal foils with glass fiber paper Tissuglas.

The first type is aluminum or copper foil coated with Teflon. The Teflon coating is from 0.00006 to 0.005 in. thick on a foil 0.0002 to 0.005 in. thick. The primary uses for this type are expected to be for electrical equipment and as a mold release or as a very thin high temperature gasketing.

The second and very latest type is a laminate—the thinnest and most flexible printed circuit material now available with an operating temperature of over 300 F. One kind of laminate consists of one mil thickness of AMF's Tissuglas (matted submicron glass fibers) and ¾ mil thickness copper with 400 volts dielectric strength.

Using this ultra-thin flexible printed circuit material, Micro Systems, Inc., a subsidiary of Electro-Optical Systems, Inc., produced an ultrasensitive strain detection device. The device is applicable to measurement of minute strain imposed upon small areas. Amfoil makes possible microstrain measurements upon very small areas with a sensitivity 50 to 60 times that of metallic counterparts. Of special value is the stability of Amfoil over the entire operating range of the "Micro-Sensor," −65 F to +180 F.

Material for Superconductive Magnets.—Scientists of the Radio Corporation of America have developed a new mass-production process that opens the way for the first time to widespread practical use of extremely simple superconductive magnets, using no power, to generate enormous magnetic fields for large nuclear research machines and for ultrasensitive receivers used in radar, radio astronomy, and space communications.

The new process was described by RCA research executives as a technical advance that may rival in importance the development of mass-production techniques for transistors.

The RCA development is a simple chemical method for rapid and continuous growth of crystalline niobium-tin, a compound superconducting material recently found to possess an ability to generate and sustain very strong magnetic fields without any power dissipation. This means that magnets made with the material will continue to operate indefinitely without consuming any power except for a small initial voltage to start a current flowing.

Until now, no satisfactory method has been devised to produce the material in the desired crystal form in quantities necessary for widespread use, largely because of the extreme brittleness of the crystalline substance and the consequent difficulty of working with it. The RCA laboratory apparatus is capable of producing uniform crystal coatings of niobium-tin on a fine wire at the rate of 30 feet per hour. Production refinements are expected to greatly increase this rate.

According to the RCA scientists, the finished wire has performed with "outstanding" success in tests involving as many as ninety turns of wire around a ¾-in. diameter coil form. In addition, tests on short-wire lengths performed at the Lawrence Radiation Laboratory, University of California, Berkeley, indicate that the wire remains superconductive in 94,000 gauss magnetic fields, while supporting a current of 7 amperes. This represents a current density in the superconductive coating of 100,000 amperes per square centimeter. These tests demonstrate the feasibility of winding the wire in any desired lengths to form extremely powerful magnets without the danger of its cracking or otherwise

losing its useful properties. The RCA scientists added that the same basic process has been used to produce hollow tubes and thin films of crystalline niobium-tin for possible application in new types of electronic devices for communications, space, and computer functions.

The new mass-production process was developed at RCA's David Sarnoff Research Center, Princeton, N. J., by a research team including the coinventors, Dr. Joseph J. Hanak and Mr. John L. Cooper, and Dr. George Cody, working under the direction of Dr. Fred D. Rosi, Associate Laboratory Director, Materials Research Laboratory. During recent phases of the project, the group has been aided by Kurt Strater and Lewis Gnau, of the engineering staff at the RCA Electron Tube Division, Harrison, N. J.

The scientists gave the following data on the laboratory work and its significance:

The importance of the niobium-tin compound lies in its unusual properties as a superconductor—a type of material that loses its electrical resistance at extremely low temperatures, close to absolute zero, so that it will continue to carry indefinitely a flow of current that is started by a small initial pulse. This superconductive property disappears in all previously known materials with a relatively small increase of temperature or in the presence of a strong magnetic field.

Niobium-tin, however, has been found to remain superconductive at somewhat higher temperature than do any other known superconductors, and—what is most important—in far stronger magnetic fields. For this reason, it has been regarded as a possible source of extremely high magnetic fields that can be sustained without consuming very large amounts of power.

Present electromagnets operate typically by inducing a magnetic field in an iron core by means of an electric current passing through wire wound around the core. To obtain very strong fields, such as those needed for cyclotrons and other large nuclear research machines, this technique requires equipment weighing tens or hundreds of tons, drawing continuously on a power supply of hundreds of thousands or millions of watts.

A superconductive magnet, wound with niobium-tin wire, or made with a thin film of the material, is expected to produce similar very strong magnetic fields with no input of power except for the small initial pulse required to start the current flowing. Because of the special property of the niobium-tin compound, it will remain superconductive even when the magnetic field rises to a very high level.

The RCA scientists pointed out that these characteristics of niobium-tin have been known for some time, but that widespread application of the material has been delayed by the absence of a suitable process for making the wire in quantity.

Microetch Varactor Diodes.—The Philco Precision-Etch processing technique which made possible MADT transistors has now been applied in the development of a new generation of varactor diodes designed as highly efficient harmonic generators.

Dr. C. H. Sutcliffe, general manager of the Special Products Operation, Lansdale Division, described Philco's new line of microetch germanium varactors which provide simultaneous operation at maximum frequency and maximum voltage.

Dr. Sutcliffe stated that the new series, comprising types L-4110, L-

4111, and L-4112, have very high cutoff frequency and greatly increased capacitance variation promoting unusual efficiency.

According to Dr. Sutcliffe, the performance of these varactors as harmonic generator devices was continually monitored in Philco harmonic generator circuits. 1 Gc to 2 Gc and 2 Gc to 4 Gc doublers of the shunt and series type were employed, as were quadruplers from 600 mc to 2.4 Gc and quintuplers from 600 mc to 3 Gc. At still higher frequencies, evaluations were carried out in X-band quadruplers. For these various types of harmonic generator circuits, typical efficiencies ranged from 35 to 60 per cent in the *S*-band multipliers while an efficiency of 15 per cent was observed for the X-band quadrupler. The data observed in these several microwave harmonic generator circuits, he said, together with microwave impedance measurement data, provided a comprehensive understanding of varactor design requirements for use in harmonic generator circuits.

The abrupt junction nature of the devices, he said, together with the low contact potential inherent in germanium, accounts for their large capacitance variation over the operating voltage range.

In the microetch varactors, the ratio of zero bias capacitance to optimum bias point capacitance is six to one, typically.

Germanium was Philco's choice for material among current practical semiconductors since the MADT Precision-Etch technique permits the attainment of minimum series resistance with germanium material through the utilization of semiconductor membrane thicknesses as narrow as 3×10^{-5} in.

Standard Philco jet etching techniques allow reliable control of the desired microwave structure, he said. Infrared transmission through the etched membrane is measured and used to precisely control the etching action. The junction electrode is electrochemically jet-plated with precise alignment to locate it in the center portion of the etched area. (This guarantees utilization of the thinnest portion of the membrane, and, consequently, achieves the lowest value of series resistance.) The junction size is determined by the size of the plated metallic electrode.

The three new devices, in order of ascending cutoff frequency, were designed primarily as efficient harmonic generators at frequencies up to 3 Gc, 6 Gc, and 10 Gc. They may also be used as parametric amplifiers or rf tuning devices.

Dr. Sutcliffe indicated that this first series of germanium microetch varactors will be expanded by the addition of several newer members designed for harmonic outputs at X-band and above. Cutoff frequencies will range from 150 to 200 Gc.

Transcontinental Combat Logistics Network.—Installation of the U. S. Air Force Combat Logistics Network (COMLOGNET), a major advance in transmitting logistic data and message communications, has been announced by Major General Harold W. Grant, Commander of the Air Force Communications Service (AFCS).

Western Union is prime contractor for the mammoth project, with Radio Corporation of America the supplier of the Automatic Electronic Switching Center equipment.

The Norton Air Force Base installation, at San Bernardino, Calif., is the first of five main centers through which data and messages from more than 350 air bases, depots, and stations in the United States will be

interchanged. Completely automatic and compatible with other Department of Defense communications systems, the COMLOGNET network will be capable of sending and receiving more than one hundred million words, or an equivalent seven million punched cards daily.

The first shipment of equipment including RCA magnetic tape storage units, computer and switching components as well as Western Union technical control components and terminal equipment racks was delivered directly from Camden, New Jersey and Chattanooga, Tennessee, where pre-shipment operating tests were conducted by the two companies.

When completed in 1962, COMLOGNET will be operated by Air Force Communications Service personnel especially trained in the advanced telecommunications techniques developed through eight years of research by the Air Force and Western Union.

Centers similar to the one at Norton Air Force Base will be established at McClellan Air Force Base, Sacramento, Calif.; Wright Patterson Air Force Base, Dayton, Ohio; Tinker Air Force Base, Oklahoma City, and Andrews Air Force Base, Maryland.

Each of the five centers will contain both message and circuit switching elements, each able to operate independently. They will integrate data processing with record communications techniques through automatic electronic means requiring a minimum of human attendance.

A pioneer achievement in communications, COMLOGNET is a major step in the development of space-age worldwide communications. Information can be fed into the system in punch-card, paper tape, magnetic tape or printed page form. All control information required for the switching centers to automatically perform switching, routing, and other processing functions from origin to destination are contained in the message headings. These include several levels of priority with the two highest levels able to automatically pre-empt the circuits in order to expedite transmission at any time.

The RCA-developed computers used at the centers employ core, drum and magnetic tape memory storage. A permanent record of all traffic is registered automatically for later reference.

Upon completion, the COMLOGNET system will be the world's largest and most advanced data communication system, attaining a degree of speed, accuracy and reliability never before achieved. It will utilize a wide range of equipment to transmit information at speeds from one hundred to three thousand words per minute. The centers are designed to employ speeds twenty times that rate when needed and are built on a modular basis permitting future growth and adaptation to advances in data communications.

Universal Transistor.—A new silicon transistor that can perform the jobs of up to 40 per cent of the more than 2000 transistor types now on the market has been developed by the Radio Corporation of America.

RCA described the development as a "significant step" in the search for a universal transistor—one that can handle every combination of power, frequency and energy required to operate modern electronic equipment.

The new unit is the first in the industry adaptable to a wide variety of uses in the complex electronic circuits of military weapons and communications systems, industrial control devices, data processing equip-

ment, and high quality consumer products, according to T. R. Hays, Marketing Manager, RCA Semiconductor & Materials Division.

Mr. Hays said the search for a universal transistor was prompted by the "astonishing growth" in the number and kinds of transistors made.

"This has created inevitable headaches for equipment manufacturers in terms of purchasing, maintaining adequate and sufficiently diversified inventories, and in selecting transistors best suited to a given equipment design," Mr. Hays stated.

Therefore, he declared, "RCA's development of an all-round workhorse device capable of high performance over an extended sweep of applications and circuit requirements is of first rate importance."

The new 2N2102 transistor owes its virtuoso performance to a newly perfected manufacturing technology, Mr. Hays disclosed.

"We have succeeded in wedding triple diffusion and planar manufacturing techniques, both relatively new and highly exacting processes, in a single transistor. Along with other techniques, they have combined to give the 2N2102 phenomenal powers for amplifying, switching or regulating current flow—the three primary areas where transistors are used. In addition, they afforded the device great electrical stability and reliability," Mr. Hays said.

Comparing the unit's versatility to that of a singer with a range of five octaves (two are considered excellent), he explained that some fifty-five electrical and mechanical tests must be conducted on the new transistor during production to insure high performance across the great breadth of applications in which it can serve. This is many more tests than required on standard products, he observed.

Recorder that Monitors up to 40 Machines.—A desk-top size device that monitors and records the efficiency of utilization of up to 40 production or data processing machines is now being manufactured and marketed by Electronic Associates, Inc. Known as the Bar Chart Recorder, the unit operates remotely from production areas and provides cost and production control information on a continuous bar chart.

Dr. John Gorrell, who developed the unit, said the recorder provides management with an accurate graphic picture of each machine's productive and nonproductive effort. It can also provide such information as the operator number, job and shift number corresponding to any manual effort on the part of the operator.

Every 20 seconds the recorder samples and graphically records each machine's performance. If the machine is operating, the fact is recorded with a short horizontal line in the appropriate bar or channel area. Lines do not appear if the machine is idle. Recorder is available with charting time of 9, 17 or 24 hours per chart.

Basic units of the recorder are a rotating drum, an inkless chart with 40 vertical bars or channels, and a stylus. The recorder is connected to the machine by telephone-type wires.

Dr. Gorrell said the minute-by-minute monitoring permits management to pinpoint lost time due to breakdowns, poor scheduling, tardiness in starting and unnecessary absences from the machine. He also said the Bar Chart Recorder can monitor almost any machine, process, operation or series of events.

Overtones of Beams of Light.–Using a new light source called a LASER, four University of Michigan physicists have for the first time produced an

overtone of a beam of light. Besides being significant as a scientific achievement, the result holds important implications for communications and other applications, the four say. The experiment was performed by Drs. Peter Franken, Wilbur Peters, and Gabriel Weinreich of the U-M physics faculty and by Alan Hill, an undergraduate. They report their results in a recent issue of the American Physical Society's *Physical Review Letters*.

They produced the second harmonic (or overtone) of red light by focusing the LASER'S intense beam of pure red light into a quartz crystal. The second harmonic—a deep blue beam—was produced as a result of the extremely high intensity of the red light at this focus. (For comparison, the focussed intensity of the LASER light is one billion times greater than sunlight on the earth's surface.)

In the crystal, this intensity produces the blue light from the red in much the same way overtones are produced in music, the physicists point out. If you twang a guitar string with great force, you get more intense overtones than if you strum it softly.

The LASER beam was monochromatic (or "pure" color) at 7000 Angstroms. The blue beam, an almost invisible violet which was considerably weaker than the LASER beam, was monochromatic at 3500 Angstroms.

To produce the intense beam, the U-M physicists use a LASER purchased from Trion Instruments, Inc., an Ann Arbor firm. They termed this LASER "one of the most powerful in the world."

The beam produces a power density of 100 million watts per square centimeter at the focus. (By contrast, the intensity of sunlight at the earth's surface is one-tenth of a watt per square centimeter.)

"This is a striking example of optical electronics," they said. "Although the production of the second harmonic is now relatively weak, the possibilities for unique devices based on this effect merit intense investigation."

Light waves are of the same substance as radio (electromagnetic) waves, but are of different frequency and wavelength—far shorter, for example, than the shortest radio waves now in use for communications.

One of the implications of the experiment is that it may open the way for using electromagnetic waves in the frequency region of light for communication purposes.

For this, the light beam must be modulated (or mixed) as are radio waves—to carry a signal. The mixing provides oscillations in the waves, the oscillations varying according to the mixing—the way the message is put on the beam.

If light waves can be used for this—and the U-M physicists point out that there are many difficult problems to be solved—beams of light would be constructed with a message carrying capacity many times that of the longer radio waves.

The Michigan scientists point out that—if these problems can be solved—a single light beam could carry all the messages that now go across country on regular communication channels.

Another application of this achievement is to permit the investigation of materials' properties that scientists haven't been able to study before—for example, the non-linear optical properties of various materials.

The U-M group is now collecting various other materials at the rate of two or three per day to test them for their non-linear optical properties and to try to produce other overtones of light. Crystals of these materials are

being supplied by the Clevite Corp. of Cleveland.

The LASER was invented about a year ago. "LASER" is an acronym for "light amplification by stimulated emission of radiation." Since the initial announcement of successful LASER development by Hughes Aircraft and Bell Laboratories, several groups have developed their own LASER.

Simultaneous Gas Chromatography and Radioactivity Measuring System.—Radiation Equipment & Accessories Corp. of Lynbrook, N. Y., announces availability of the E-180 Actichromagraph, a simultaneous gas chromatography and radioactivity measuring system.

Gas chromatography is an extremely useful analytical method which sharply separates components of complex mixtures. Authorities in this field say that before long one can reasonably expect that this method will be used not only as an analytical tool but also as a method of preparing ultrapure materials. If the process can be made continuous, it should be able to provide laboratory chemicals and even commercial compounds having a totally new order of purity.

The E-180 system consists of a gas-liquid chromatograph and a special high temperature ionization chamber for the measurement of tritium and carbon-14. This combination in a single instrument allows the convenient and complete separation of many classes of substances and the simultaneous measurement of the amounts of tagged radioactivity in each component, providing a powerful tool for the research laboratory and an appropriate system for the teaching institution.

Major features include: wide selection of stationary phase columns; differential flow rate of gas; transistorized power supplies, read-out and control circuitry; continuous reading flow meter; precision soap-film meter for calibration; molecular sieve filter for cleaning carrier gas; two outputs for standard recorders.

Light Stabilized Medium Impact Polystyrene.—Development of medium impact grade of light stabilized polystyrene that opens new applications in the lighting field has been announced by The Dow Chemical Company, of Midland, Mich.

Marketed as Styron Verelite 374, the medium impact polystyrene displays good toughness that will help eliminate the problems of fixture breakage during shipment and cleaning operations.

The new Verelite has excellent color stability under fluorescent light radiation and offers marked resistance to yellowing, Dow said. Stabilizers incorporated in the formulation retard discoloration caused by ultraviolet light and heat.

According to Dow development engineers, Styron Verelite 374 has 20 per cent elongation and Izod Impact Strength of 0.5 foot pound per inch of notch for a compression molded specimen. The high elongation properties allow the vacuum forming of a wide range of lighting pans.

Dow also reported superior processibility in extruders and injection molding machinery. As a result of the broad development program, light diffusers, fixture and grids are expected to provide many new applications for Styron Verelite 374, said Dow. Improved weatherability of the new tailored material also offers application for short life outdoor signs with up to two years life duration.

The medium impact light stabilized polystyrene is manufactured in a

range of granulations and colors to meet the needs of extruders and injection molders in the lighting industry, according to Dow.

The newest addition broadens to 22 the number of polystyrene formulations manufactured by The Dow Chemical Company, which pioneered the commercial development of this plastic in 1938.

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